THERMODYNAMICS AND FLUID MECHANICS 2

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FLUID MECHANICS NOTES

Boundary Layer flows

Lift & Drag

Dimensional Analysis

Turbomachinery

Compressible flow

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BOUNDARY LAYERS

INTRODUCTION

In this section we are looking at external flows around bodies or objects fully immersed in a fluid stream. As with the internal flows of Thermofluids 1, there will be viscous effects and with external flows these occur near the body and in its wake. Far from the body, often the flow can be treated as inviscid.

With internal flows, as the boundary layers grow thicker they eventually meet in the middle of the pipe/duct. With external flows the flow is unconfined and the boundary layer can continue to grow no matter how thick they become. There are many engineering situations that require the study of external flows:

- Aeroplanes, rockets, projectiles (aerodynamics)
- Ships, submarines, torpedoes (hydrodynamics)
- Cars, lorries, trains etc
- Buildings, bridges
- Moored platforms, buoys, cables, pilings, breakwaters (ocean-related)

There are essentially three techniques used to study external flows and these are:

(i) Numerical methods (eg computational fluid dynamics, CFD)
(ii) Experimentation
(iii) Boundary layer theory

CFD is a huge field and with ever increasing computational capacity becoming affordable it is now possible to investigate a huge range of practical engineering problems using this approach. Conventional CFD still does not cope well with transition from laminar to turbulent transitional flow although direct numerical simulation (DNS) and large eddy simulation (LES) methods are somewhat bridging the gap. CFD is not studied as part of this module.

Experimentation remains a common method despite the inherent cost and difficulties in measuring non-intrusively (or minimizing such effects). Dimensional analysis plays a significant role in supporting intelligent data analysis and is studied later in this module.

Boundary layer theory allows us to get some theoretical basis for understanding and predicting boundary layer flows. The full fluid flow equations (the Navier-Stokes equations, derived later in the module) are simplified for the boundary layer such that they can be solved (in general the N-S equations are not amenable to anything other than numerical solution, hence the prevalence of CFD methods). The boundary layer solution can then be "patched" onto the far-field, inviscid flow. Boundary layer theory can be used to predict flow separation but is not applicable in the low pressure wake regions subsequently created.

BOUNDARY LAYER DESCRIPTION

Consider the case of a uniform flow stream of velocity U moving parallel to a sharp flat plate of length L. The Reynolds number is defined as $Re = \frac{\rho U L}{\mu}$. Where the Reynolds number is very low the viscous region is broad, extending ahead of the plate as well as to the sides, as illustrated in Figure 1.

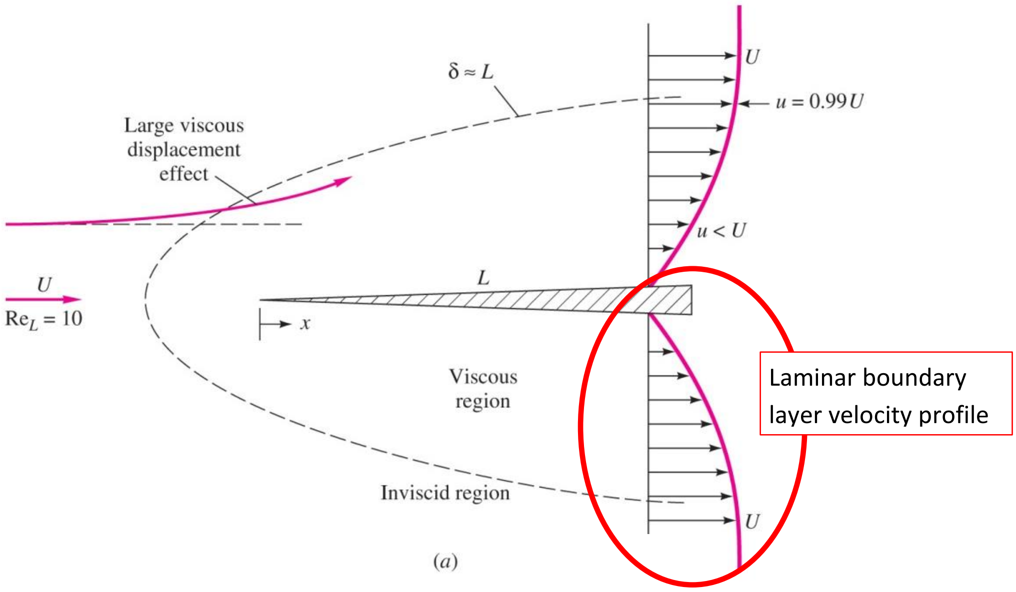

Figure 1: Low Reynolds number flow past a sharp flat plate [1]

We can define a boundary layer thickness, δ, which marks the point where the velocity parallel to the plate is 99% of the free stream velocity, and as Figure 1 shows, this region is large for low Reynolds number flow.

In contrast, when the Reynolds number is large, the boundary layer is initially laminar but transitions to turbulent (notwithstanding the laminar sublayer of course) at some distance from the front of the plate. The boundary layers are very thin and thus there is only a small overall displacement effect on the free stream. This type of flow is illustrated in Figure 2. Such flows are also far more likely to occur in most practical engineering applications. For example, at 500 miles per hour, for a 2m plate moving in atmospheric air the Reynolds number is 3×10^7.

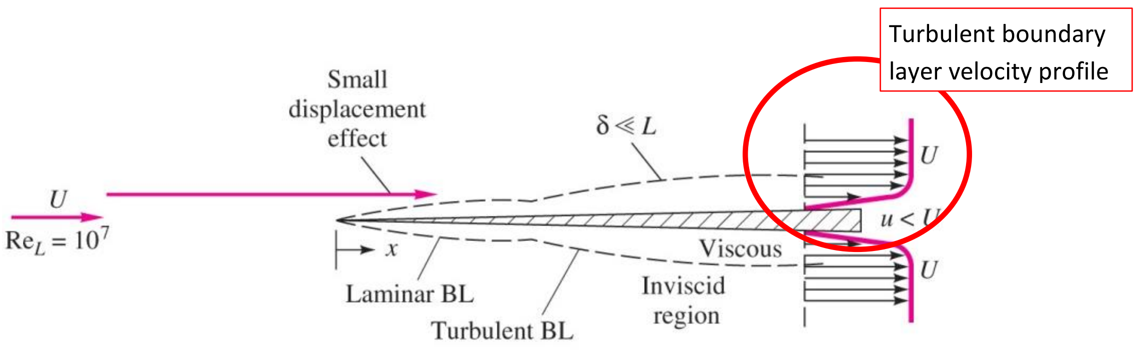

Figure 2: High Reynolds number flow past a sharp flat plate [1]

BOUNDARY LAYER THICKNESS

Because the velocity of a boundary layer approaches the free stream velocity asymptotically, the thickness is defined according to a chosen convention. The thickness δ mentioned above is based on the boundary layer velocity achieving 99% of free stream velocity. There are two other standard methods of defining the boundary layer thickness that are commonly used and these are covered in this section.

(i) Displacement thickness, δ^*

This definition considers the reduction of volume flowrate caused by the boundary layer. The flow per unit width (into the page) through a small element of thickness δy is $u\,\delta y$. If there was no boundary layer the flow through the same element would have been $u_m\,\delta y$. The reduction in volume flowrate caused by the boundary layer is therefore the shaded area in Figure 3a. If you displaced the actual surface by a distance δ^*, with no boundary layer then the same reduction in volume flowrate is found as illustrated in Figure 3b.

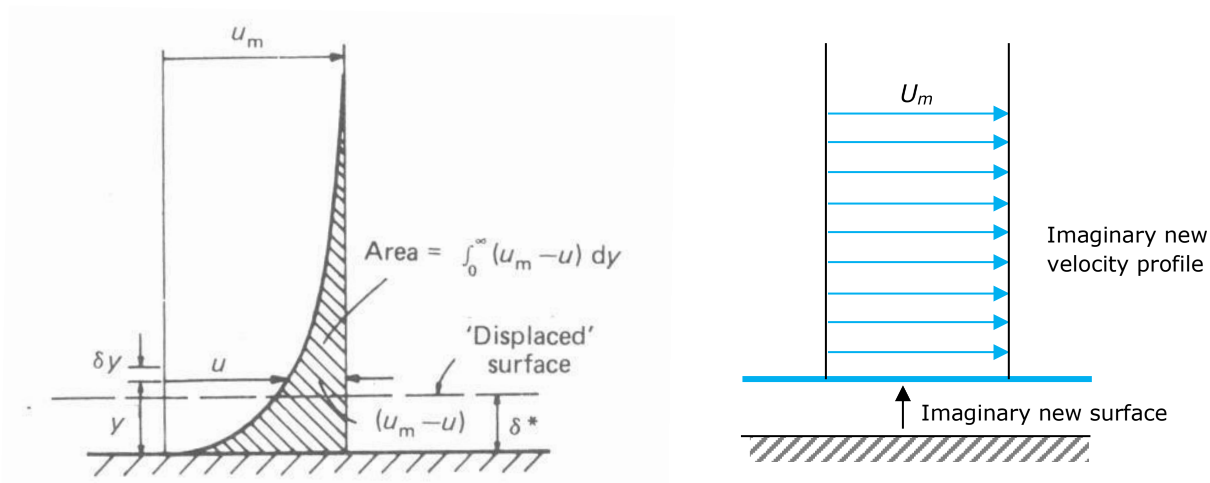

Figure 3: Displacement thickness, δ^*
a) Boundary layer profile [2], b) equivalent flow

Mathematically the displacement thickness is:

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_m}\right) dy$$

Eq 1

(ii) Momentum thickness, θ

Similar to the displacement thickness, in this case the momentum reduction is considered rather than the volume flowrate. The momentum of the fluid passing through the small element show in Figure 3 is $(\rho u\,\delta y)u$ per unit width, whereas without the boundary layer the momentum (of the same mass of fluid) would have been $(\rho u\,\delta y)u_m$. The total loss of momentum is $\int_0^\infty \rho(u_m - u)\,u\,dy$ and this is equated to the momentum of a volume of frictionless fluid of thickness θ, $(\rho u_m \theta)u_m$. Thus the momentum thickness is shown to be:

$$\theta = \int_0^\infty \frac{u}{u_m}\left(1 - \frac{u}{u_m}\right) dy$$

Eq 2

The ratio of the displacement thickness to the momentum thickness is called the *shape factor, H,*

$$H = \frac{\delta^*}{\theta}$$

Eq 3

FLAT PLATE BOUNDARY LAYER

SKIN FRICTION AND DRAG

As fluid flows past a flat plate a shear force is exerted on it due to friction between the surface of the body and the fluid. There are two ways of approaching this, one from the point of view of the fluid and the other from the point of view of the shear stress on the plate.

By applying the linear momentum equations introduced in Thermofluids 1 the force on the fluid can be investigated further.

Recall that the force exerted on a fluid in the x-direction is equal to its rate of change of momentum in the x-direction:

$$\sum F_x = \frac{d}{dt}(mu)$$

Eq 4

And that under steady conditions (no change of momentum within the control volume) this becomes:

$$\sum F_x = \sum (\dot{m}_i u_i)_{out} - \sum (\dot{m}_i u_i)_{in}$$

Eq 5

Or in words, "the net force on a control volume in the x-direction is equal to the sum of the outlet momentum fluxes minus the sum of the inlet momentum fluxes" [1]

Consider the boundary layer shown in Figure 4. Upstream the flow is uniform at velocity U_o with a boundary layer developing along the length of the plate. Line 2 is a stream line just outside the boundary layer (remember flow is wholly along a streamline so there will be no flow across line 2). We assume there is no external pressure gradient and so the only force acting on the control volume created by lines 1-4 is from the change in momentum.

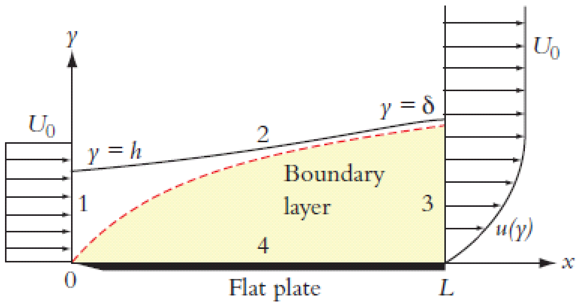

Figure 4: Boundary layer on a flat plate with appropriate control volume [3]

Momentum of fluid entering at 1 (for width *b* into page) is: $(\rho U_o hb)U_o$

Momentum of fluid leaving at 2 is: $\int_0^\delta (\rho u b dy)\, u$

The drag force acting on one side of the plate, D, is therefore:

$$F_x = -D = \int_0^\delta \rho b u^2 dy - \rho h b U_o^2$$

Eq 6

Height *h* is not known, but we can apply continuity between 1 and 2 as there is no flow across 2 or 4.

This tells us that the mass flow entering the control volume is equal to the mass flow leaving it:

$$\rho h b U_o = \int_0^\delta \rho b u dy$$

So
$$h = \int_0^\delta \frac{u}{U_o} dy$$

Thus
$$D = \rho b \int_0^\delta u(U_o - u) dy$$

Eq 7

This equation was first derived by Kármán in 1921 who further noted that as the momentum thickness is given by Eq 2, if we neglect the difference between integrating to infinity and integrating to δ then we can re-write the equation for D as:

$$D = \rho b U_o^2 \theta$$

Eq 8

Momentum thickness can thus be seen as a measure of total plate drag. Of course this drag force arises because of shear between fluid and plate at the interface and so the drag force is the summation (integral) of the shear over the plate surface. Thus:

$$D = b \int_0^x \tau_w \, dx$$

And differentiating both sides with respect to x give us:

$$\frac{dD}{dx} = b\tau_w$$

But we also know (from Eq 8) that $\frac{dD}{dx} = \rho b U_o^2 \frac{d\theta}{dx}$

So
$$\tau_w = \rho U_o^2 \frac{d\theta}{dx}$$

Eq 9

Note that this is the value of wall shear stress, τ_w existing at distance x from the start of the plate. It is a point value.

This equation is valid for both laminar and turbulent flow. Shear stress (skin friction) is often expressed as a skin friction coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_o^2}$$

Eq 10

And combining these two gives us:

$$C_f = 2\frac{d\theta}{dx}$$

Eq 11

LAMINAR BOUNDARY LAYER

Whether the far field flow is laminar or turbulent, the first part of the boundary layer will be laminar. For a flat plate the laminar part of the boundary layer is often short (and therefore negligible) but there are situations where laminar boundary layers are important. A German engineer (PRH Blasius, 1883-1970) obtained analytical equations for laminar boundary layer flow. Full derivation of the Blasius equations can be found in [1] and [2]. The equations can be solved numerically to yield the laminar boundary layer thickness:

$$\frac{\delta}{x} \approx \frac{5}{Re_x^{0.5}}$$

Eq 12

Re_x is the Reynolds number based on x, the distance along the plate from the start:

$$Re_x = \frac{\rho U_o x}{\mu}$$

For boundary layer flows transition to turbulence can occur over a range of boundary layer Reynolds numbers. For typical engineering surfaces in atmospheric type (gusty!) free streams an accepted value is:

$$Re_{x,trans} = 5 \times 10^5 \text{ [1].}$$

Eq 12 is generally taken to be valid for the range $10^3 < Re_x < 10^6$.

The velocity profile in a laminar boundary layer is approximately parabolic, having the form:

$$\frac{u}{U_m} = \frac{y}{\delta}\left(2 - \frac{y}{\delta}\right)$$

Eq 13

Figure 5 shows a comparison of this approximate parabolic profile with the exact Blasius profile, where it is seen that the parabolic approximation is very close indeed.

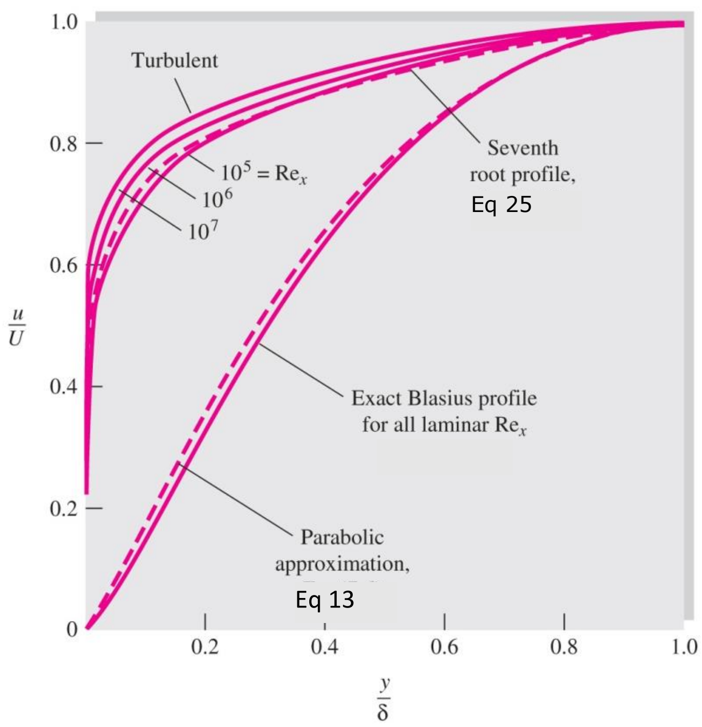

Figure 5: Dimensionless flat plate velocity profiles, laminar and turbulent [1]

Worked Example 1

A long thin flat plate is held parallel to a stream of water (take ρ=1000kg/m^3 and μ=0.001) moving at 0.1 m/s. How thick is the boundary layer at a distance of 2m from the leading edge of the plate?

Assuming transition to a turbulent boundary layer occurs at a Reynolds number of 10^6 how far along the plate does this occur?

Answers: 22 mm, 10m

Using Blasius solution for the exact velocity profile in a laminar boundary layer the wall shear and displacement thickness can be calculated and these are:

$$C_f = \frac{0.664}{Re_x^{0.5}}$$

Eq 14

And
$$\frac{\delta^*}{x} = \frac{1.721}{Re_x^{0.5}}$$

Eq 15

The momentum thickness can also be computed and is found to be:

$$\frac{\theta}{x} = \frac{0.664}{Re_x^{1/2}}$$

Eq 16

Thus the shape factor (Eq 3) for laminar flow is: $H = \frac{\delta^*}{\theta} = \frac{1.721}{0.664} = 2.59$

The drag coefficient for a flat plate is defined as:

$$C_D = \frac{D/_{bL}}{\frac{1}{2}\rho U_0^2}$$

Eq 17

And is found to be:
$$C_D = \frac{1.328}{Re_L^{0.5}}$$

Eq 18

The drag coefficient for a flat plate is twice the value of the skin friction coefficient at the trailing edge (noting that D is the drag force on one side of the plate).

Worked Example 2

A thin flat plate 20m wide by 5 m long is held parallel in a stream of air (take ρ=1.2 kg/m^3 and μ=1.8x10^{-5} kg/ms) moving at 0.8 m/s.

What is the total drag force exerted on the plate (ie both sides of the plate) by the air?

Answer: 0.2N

TURBULENT BOUNDARY LAYER

In turbulent flow near a wall or surface the shear stress is, in general, made up from two parts, the laminar shear stress and the turbulent shear stress:

$$\tau = \tau_{lam} + \tau_{turb} = \mu\frac{\partial u}{\partial y} + \rho u'v'$$

Where u' and v' are the fluctuating parts of velocities u and v (remembering that for turbulent flow each velocity component has a mean velocity component plus a random fluctuating component, ie $u = \bar{u} + u'$

A typical turbulent velocity profile is shown in Figure 6. In the near wall layer viscous shear dominates and in the outer layer turbulent shear dominates. In the overlap layer both types of shear are present.

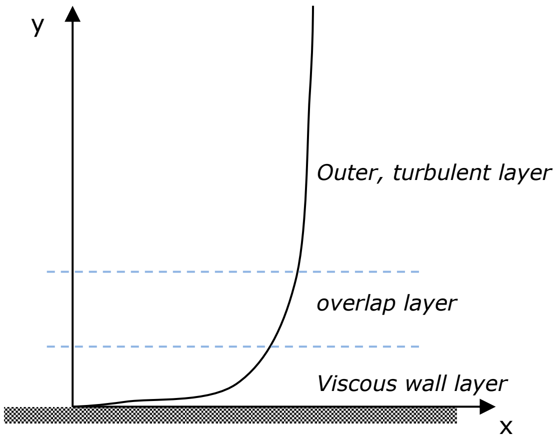

Figure 6: Velocity distribution in turbulent boundary layer

Dimensional reasoning (by Prandtl and Kármán) together with experimental data have led to a single graph that represents turbulent wall or boundary layer flow. This graph is often referred to as the law of the wall.

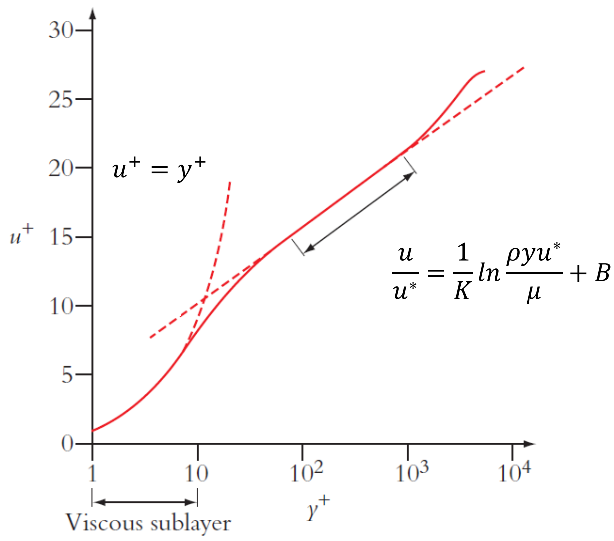

Figure 7: Velocity relationships within turbulent boundary layers

The x-axis of this graph is logarithmic and plots y$^+$, a non-dimensional distance perpendicular to the surface:

$$y^+ = \frac{\rho y u^*}{\mu}$$

Eq 19

Where
$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

Eq 20

and u* is called the friction velocity (because it has units of m/s, not because it is a velocity!).

The y-axis of the graph is u^+, a non-dimensional velocity where

$$u^+ = \frac{u}{u^*}$$

Eq 21

In the viscous sublayer $$\boldsymbol{u^+ = y^+}$$

Eq 22

In the far outer region the so-called velocity defect law applies, where $\frac{U-u}{u^*}$ is a function of $\frac{y}{\delta}$. Different curves are found here depending on whether it is pipe flow, flat plate flow etc and whether the pressure is increasing or decreasing. In between these two regions is an overlap region described by the formula:

$$\frac{u}{u^*} = \frac{1}{K} ln \frac{\rho y u^*}{\mu} + B$$

Eq 23

It turns out that this logarithmic region approximates almost the entire velocity profile and is (surprisingly) consistent with experimental data. Setting *K*=0.41 and *B*=5.0 approximates the full range of smooth wall turbulent flows. As the relationship is logarithmic it appears as a straight line on Figure 7.

If we neglect the laminar sublayer and apply this equation across the entire boundary layer, then substituting y=δ for the edge of the boundary layer gives us:

$$\frac{U}{u^*} = \frac{1}{K} ln \frac{\rho \delta u^*}{\mu} + B$$

Eq 24

Prandtl observed that turbulent velocity profiles approximately follow a 1/7th power law:

$$\left(\frac{u}{U}\right)_{turb} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

Eq 25

This is the dotted line shown amongst the turbulent profiles on Figure 5, where it can be seen that this is a good approximation. Using some clever approximations, Prandtl was able to integrate Eq 11 for turbulent flow, yielding:

$$\frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$$

Eq 26

Compare this to the equivalent for laminar flow Eq 12 and it is apparent that the turbulent boundary layer grows far more quickly than a laminar one.

Using the 1/7th power law approximation for the boundary layer velocity profile the displacement and momentum thicknesses can also be calculated and these are found to be:

$\delta^* = \frac{\delta}{8}$ and $\theta = \frac{7\delta}{72}$. Thus for turbulent flow $H = \frac{\delta^*}{\theta} = \frac{1/8}{7/72} = 1.3$

Following on from Eq 26 it can further be shown that:

$$C_f \approx \frac{0.027}{Re_x^{1/7}}$$

Eq 27

The drag coefficient can also be evaluated as

$$C_D = \frac{0.031}{Re_L^{1/7}}$$

Eq 28

An alternative approximation utilizes the Blasius approximation for wall shear stress in turbulent flow in smooth pipes (see [4] or [5] for derivation). This has some advantages compared to that used by Prandtl as the 1/7th power law profile of Eq 25 does not really hold at the wall. Using this approach we obtain:

$$\frac{\delta}{x} \approx \frac{0.382}{Re_x^{1/5}}$$

Eq 29

FLAT PLATE BOUNDARY LAYER SUMMARY

The quantities derived for laminar a turbulent boundary layers on a smooth flat plate are summarized in Table 1 below.

Table 1: Summary of correlations and formulae for flat plate boundary layers

Quantity	**Laminar flow**	**Turbulent flow** **Prandtl approximation**	**Turbulent flow** **Blasius pipe flow approximation**
Boundary layer thickness, δ	$\frac{\delta}{x} \approx \frac{5}{Re_x^{0.5}}$	$\frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$	$\frac{\delta}{x} \approx \frac{0.37}{Re_x^{1/5}}$
Displacement thickness, δ^*	$\frac{\delta^*}{x} = \frac{1.721}{Re_x^{0.5}}$	$\frac{\delta^*}{x} \approx \frac{0.02}{Re_x^{1/7}}$	$\frac{\delta^*}{x} \approx \frac{0.046}{Re_x^{1/5}}$
Momentum thickness, θ	$\frac{\theta}{x} = \frac{0.664}{Re_x^{1/2}}$	$\frac{\theta}{x} = \frac{0.0156}{Re_x^{1/7}}$	$\frac{\theta}{x} = \frac{0.036}{Re_x^{1/5}}$
Shape factor, H	2.59	1.28	1.28
Skin friction coefficient, C_f	$C_f = \frac{0.664}{Re_x^{0.5}}$	$C_f \approx \frac{0.027}{Re_x^{1/7}}$	$C_f \approx \frac{0.058}{Re_x^{1/5}}$
Drag coefficient, C_D	$C_D = \frac{1.328}{Re_x^{1/2}}$	$C_D = \frac{0.031}{Re_x^{1/7}}$	$C_D = \frac{0.7251}{Re_x^{1/5}}$

Worked Example 3

A thin flat long plate is held parallel in a stream of air (take ρ=1.2 kg/m^3 and μ=1.8x10^{-5} kg/ms) moving at 25 m/s. What is the percentage difference in the boundary layer momentum thickness calculated using the Blasius approximation compared to that of Prandtl at a distance 2m from the leading edge of the plate?

Answer: 2.6%

EFFECT OF ROUGHNESS

Roughness has quite a significant effect on the drag coefficient, primarily by:

- Promoting an earlier transition to turbulence
- Modifying the velocity profile of the turbulent boundary layer

The roughness parameter is $\frac{x}{\varepsilon}$ or $\frac{L}{\varepsilon}$ where ε is the mean roughness height (similar to the pipe flow equivalent where the roughness parameter is $\frac{\varepsilon}{d}$). The surface can be treated as hydraulically smooth if the Reynolds number based on friction velocity u* (Eq 20) and the roughness height is less than 5. Ie

$$\frac{\rho u^* \varepsilon}{\mu} < 5$$

Eq 30

Figure 8 shows a chart correlating drag coefficient C_D with Reynolds number and roughness parameter. The laminar and turbulent equations for C_D for smooth plates appear at the bottom of the chart with the transition zone bounded by the equations of Eq 32. Lines of increasing roughness are on the right of the chart. As can be seen, towards the top right of the diagram C_D is independent of Reynolds number and depends only on the roughness parameter. This region is described as “fully rough” and curve fitting to the data gives empirical relationships for C_f and C_D in this region:

$$C_f = \left(2.87 + 1.58 log\frac{x}{\epsilon}\right)^{-2.5}, \qquad C_D = \left(1.89 + 1.62 log\frac{L}{\epsilon}\right)^{-2.5}$$

Eq 31

$$C_D \approx \begin{cases} \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L} & Re_{trans} = 5 \times 10^5 \\ \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L} & Re_{trans} = 3 \times 10^6 \end{cases}$$

Eq 32

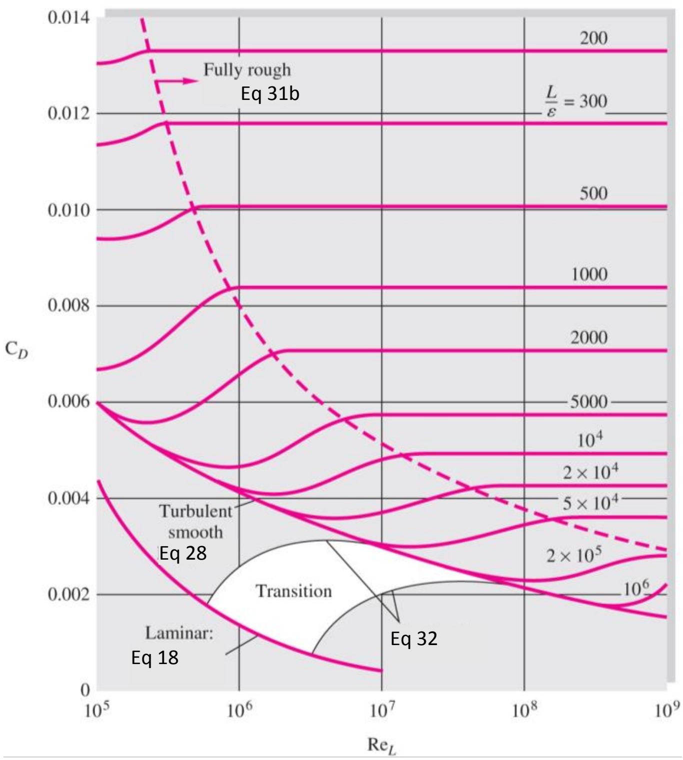

Figure 8: Drag coefficients of laminar and turbulent boundary layers on smooth and rough flat plates [1] (analogous to Moody chart for pipe flow)

It is difficult to now the drag coefficient in the transition region because the onset of transition depends on many factors. The region is bounded by the two lines shown (Eq 32) and it is usually best to assume transition along the left-hand line unless you have additional knowledge.

Worked Example 4

A hydrofoil 0.4 m long and 2m wide is placed in a sea water flow of 14 m/s (density 1020 kg/m^3 and viscosity 0.0012 kg/ms). When new the plate is smooth but after a few years barnacles have grown on the surface creating a mean roughness height of 0.5mm. By how much will the drag on the plate have increased?

Answer: 943.6N

EFFECT OF PRESSURE GRADIENT

So far we have only considered situations with a zero pressure gradient. Consider flow around a circular cylinder, as illustrated in Figure 9. Only at very low flows (called Stokes flow, or creeping flow) will the boundary layer remain attached all the way round the sphere. For Reynolds numbers greater than around 1 the boundary layer will detach and a low pressure wake region forms behind the cylinder. The boundary layer detaches because of the pressure gradient “along” the boundary layer.

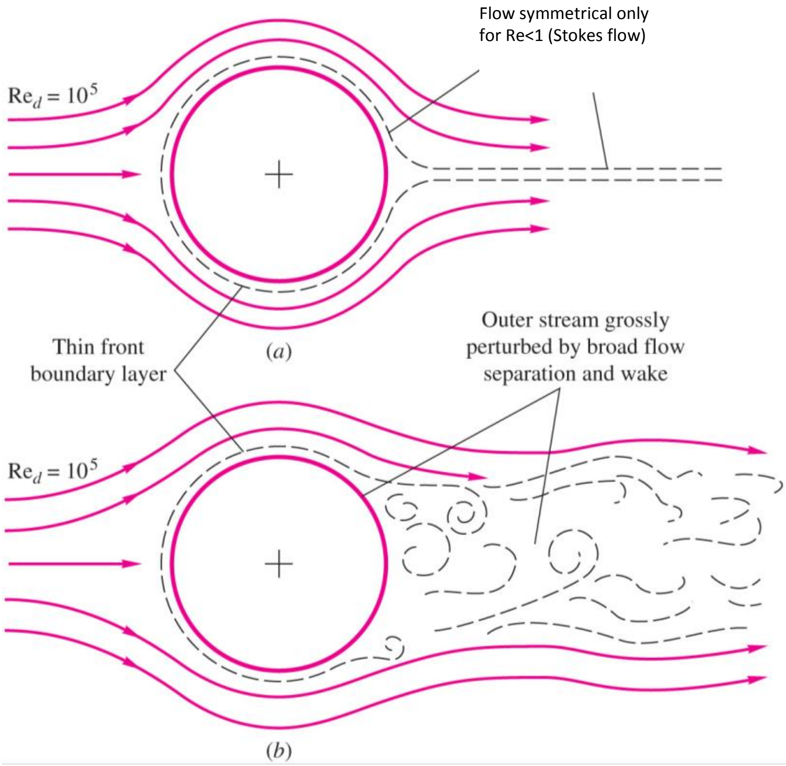

Figure 9: Flow past a circular cylinder. For Re>1 the flow separates and a low pressure wake region is formed)from [1])

A boundary layer loses kinetic energy (it slows down). In a *favourable* pressure gradient, pressure is decreasing $\left(\frac{dp}{dx} < 0,\ \frac{dU}{dx} > 0\right)$ and so the “pressure force” is acting in the direction of the flow. The boundary layer can therefore replenish its energy from the main flow and flow separation will not occur. In an *adverse* pressure gradient $\left(\frac{dp}{dx} > 0, \frac{dU}{dx} < 0\right)$ the “pressure force” is acting against the flow and so the lost energy is not replenished. This will lead to further slowing of the boundary layer near the surface and ultimately to flow separation. At the point of separation the wall shear stress goes to zero $\tau_w = 0$. This is illustrated in Figure 10.

Once the boundary layer has separated from the surface a low pressure wake region is created. Consequently there is a pressure difference across the body that contributes to the drag. There is no theory that enables quantitative calculation of the forces and flow behavior around an arbitrary body immersed in an arbitrary flow and experimentation is often the only option (although CFD is becoming more prevalent and valuable).

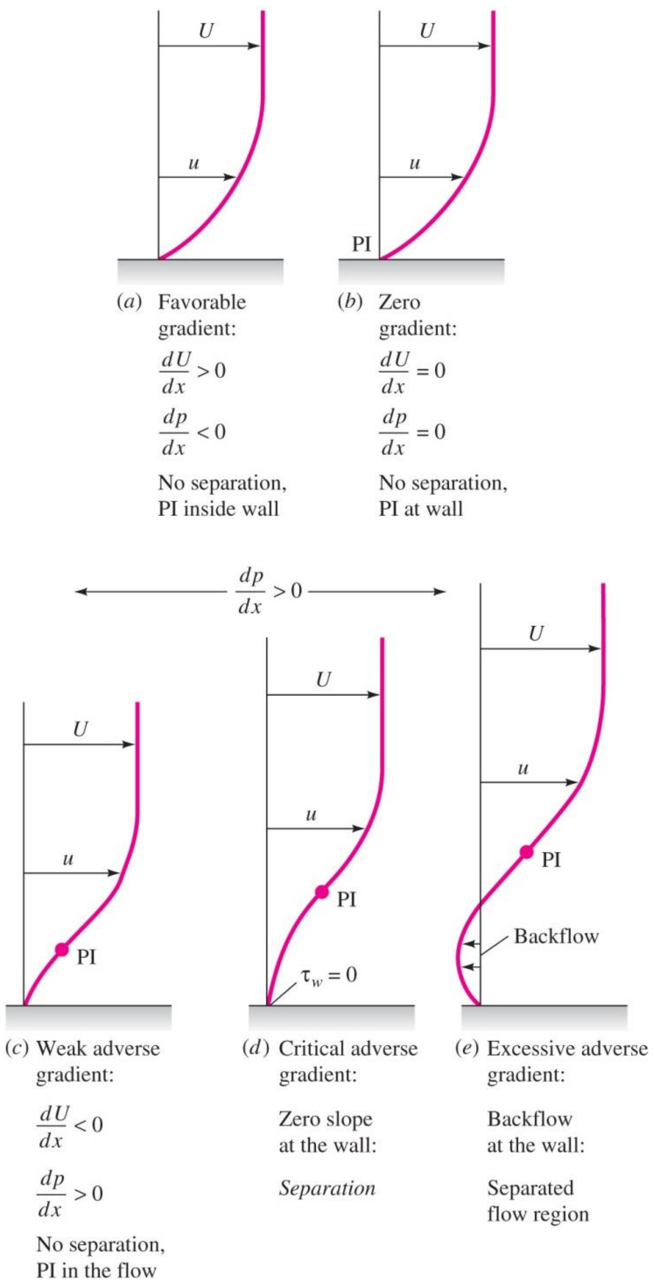

Figure 10: Effect of pressure gradient on boundary layer profiles (PI=point of inflexion)

USEFUL TEXTBOOKS

1. Frank White, Fluid Mechanics
2. B S Massey, Mechanics of Fluids
3. RW Fox, AT McDonald & PJ Pritchard, Introduction to Fluid Mechanics
4. JF Douglas, JM Gasiorek, JA Swaffield & LB Jack, Fluid Mechanics