Thermodynamics and Fluid Mechanics 2

MMME2047 UNUK

Fluids Topic 1: Navier-Stokes equations

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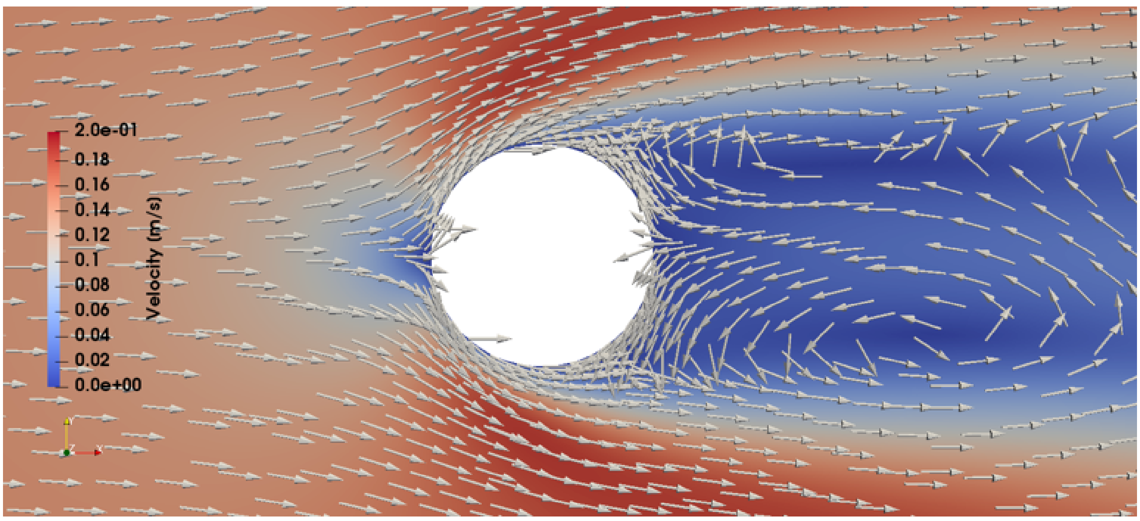


These notes provide some text to the slides for Fluids Topic 1 − Navier-Stokes equations, presented in class in the module of Thermodynamics and Fluid Mechanics 2. The present notes follow the material in Chapter 4 of the book of F. White, Fluid Mechanics, 8th Edition. The book can be found in the George Green library of the University of Nottingham, see this link. For further reading and assessment in this topic please refer to F. White book, Chapter 4 and exercises at the end of it.

Contents

1 Velocity of a fluid in motion

The velocity of a fluid in motion, see Fig. 1, can be represented as a vector field that varies in space (x, y, z) and time t, and therefore expressed as:

$$\mathbf{V}(x,y,z,t) = u(x,y,z,t)\hat{\mathbf{i}} + v(x,y,z,t)\hat{\mathbf{j}} + w(x,y,z,t)\hat{\mathbf{k}}, \tag{1}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ indicate the unit vectors in the direction of the x, y, and z axes of a three dimensional Cartesian coordinate system, and u, v, and w represent the velocity components (units: m/s) along the three coordinate directions.

But which equations do govern the flow field? In principle, we have:

- Conservation of mass
- Conservation of momentum
- Conservation of energy

However, if the flow is isothermal, the mass and momentum equations alone are sufficient to describe the flow of a fluid. In the present notes, we focus on the conservation of mass and momentum.

For simplicity, we derive the flow equations for a two-dimensional configuration, and we will extend the derivation to a three-dimensional flow afterwards. An important note: in nature, flow is always 3D; however, under certain conditions, the flow can be simplified using a 2D description. For example, if the cylinder in Fig. 1 is very long in the direction perpendicular to the page (say z), we can imagine that, far from the cylinder ends, the flow field will appear to be independent of the z location, and the velocity component w will be zero. In this case, far from the cylinder ends the velocity field will be well described with a two-dimensional vector field:

$$\mathbf{V}(x,y,t) = u(x,y,t)\hat{\mathbf{i}} + v(x,y,t)\hat{\mathbf{j}}. \tag{2}$$

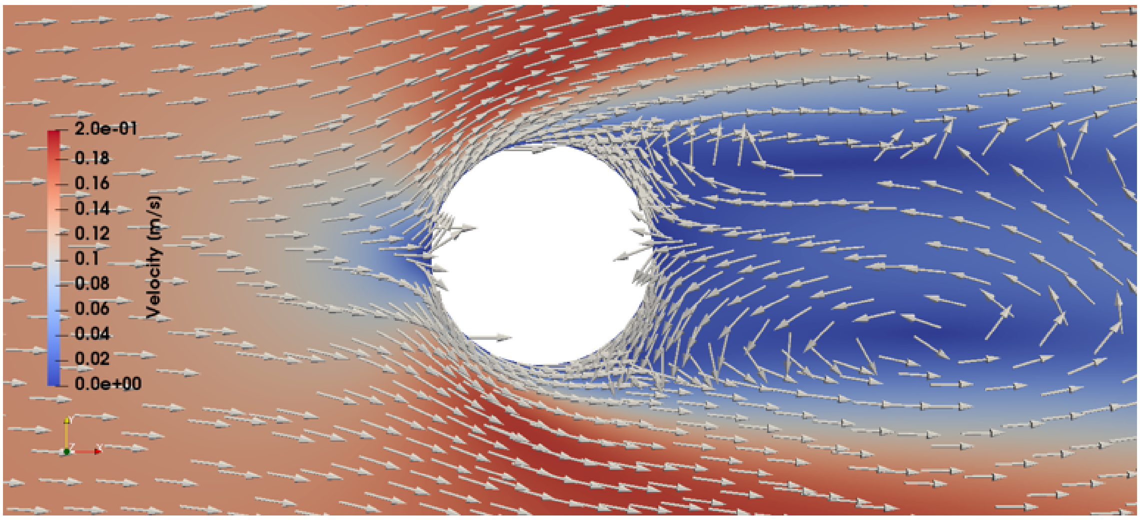


Figure 1: Velocity field for flow past a cylinder. Flow is from left to right. The velocity vectors (not to scale) indicate the direction of the flow, while the colours indicate the magnitude of the velocity field.

This is the situation that we assume as a starting point to derive the flow equations. The extension to 3D flows is trivial, although lengthy.

2 Conservation of mass

Let's start considering a generic, fixed control volume within the flow region in Fig. 1, bounded by a closed surface through which the fluid penetrates. The conservation of mass, also known as *continuity*, requires that the temporal variation of the mass in the control volume must be equal to the net inflow/outflow through the control volume boundary surface.

Let's consider a 2D configuration, and an infinitesimal fixed control volume of width dx, height dy, and reference depth 1 in the z direction, as skecthed in Fig. 2. We denote as M the mass included in the control volume (units: kg), $\dot{M}_{in}$ the mass flow rate entering the control volume through the west and south faces (units: kg/s), and $\dot{M}_{out}$ the mass flow rate exiting the control volume through the north and east faces. As stated above, continuity requires that:

$$\frac{\partial M}{\partial t} + \dot{M}_{out} - \dot{M}_{in} = 0, \tag{3}$$

which has units kg/s. We need now to express the three terms in the equation above. The mass included in the control volume is:

$$M = \int_{volume} \rho(x, y, t)\, dx\, dy, \tag{4}$$

where ρ denotes the density of the fluid, units kg/m^3, which depends on both space and time. Note that the differential in the integral should be $dx\, dy\, 1$, to account for the control volume extension along z, but the 1, that has the unit of a length, is dropped for simplicity of notation. The temporal variation of the mass in the control volume can be expressed as:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\int_{volume} \rho(x, y, t)\, dx\, dy = \int_{volume} \frac{\partial \rho}{\partial t}\, dx\, dy, \tag{5}$$

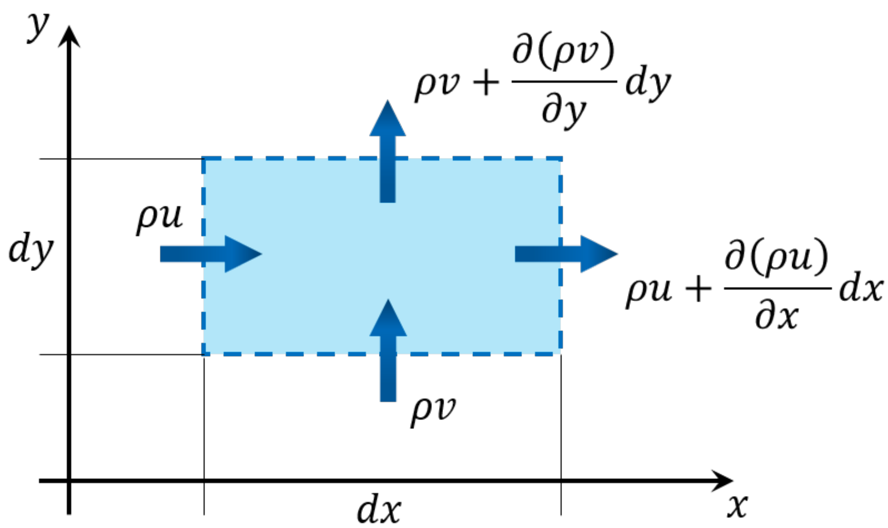


Figure 2: Control volume utilised to derive the equation governing the conservation of mass.

where the derivative enters the integral as the control volume is fixed. We assume that the control volume is so small that the volume integral reduces to a differential term:

$$\int_{volume} \frac{\partial \rho}{\partial t}\, dx\, dy \approx \frac{\partial \rho}{\partial t}\, dx\, dy. \tag{6}$$

To express $\dot{M}_{in}$, we refer to u and v as the horizontal and vertical components of the velocity on the west and south faces, respectively, so that:

$$\dot{M}_{in} = \rho u\, dy + \rho v\, dx, \tag{7}$$

where $\rho u\, dy$ is the mass flow rate through the west face and $\rho v\, dx$ the mass flow rate through the south face. Again, bear in mind that the control volume extension along z is 1. To express $\dot{M}_{out}$, we need to know the flow rates on the east and north faces, keeping in mind that ρ, u and v vary in space. To this end, we can use the first-order Taylor expansion of a generic function of two variables $f(x, y)$ along each direction:

$$f(x + dx, y) \quad = f(x, y) + \frac{\partial f}{\partial x} dx, \tag{8a}$$

$$f(x, y + dy) \quad = f(x, y) + \frac{\partial f}{\partial y} dy, \tag{8b}$$

so that on the east face $(x + dx)$ the outlet mass flow is:

$$\left[\rho u + \frac{\partial(\rho u)}{\partial x} dx\right] dy,$$

and on the north face $(y + dy)$:

$$\left[\rho v + \frac{\partial(\rho v)}{\partial y} dy\right] dx.$$

Therefore,

$$\dot{M}_{out} = \left[\rho u + \frac{\partial(\rho u)}{\partial x} dx\right] dy + \left[\rho v + \frac{\partial(\rho v)}{\partial y} dy\right] dx, \tag{9}$$

where both terms at the right-hand side should be multiplied by 1 to account for the z extension of the control volume. Now, introducing Eqs. (5)-(6), (7), and (9), in Eq. (3), and deleting the terms $\rho u\, dy$ and $\rho v\, dx$ that appear twice with opposite signs, we obtain:

$$\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right] dx\, dy = 0. \tag{10}$$

Note that the element volume is actually $dx\, dy\, 1$ (units: m^3) so that Eq. (10) is still in kg/s units, as for Eq. (3). Now, the element volume can be eliminated, thus leading to the final expression for the conservation of mass for an infinitesimal control volume:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{11}$$

which is in $kg/(m^3 s)$ units, i.e. mass per unit volume and time.

Incompressible flow

In the special case of an incompressible flow, the density changes are negligible and therefore $\partial\rho/\partial t = 0$, and ρ can be regarded as a constant. This simplifies the continuity equation as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{12}$$

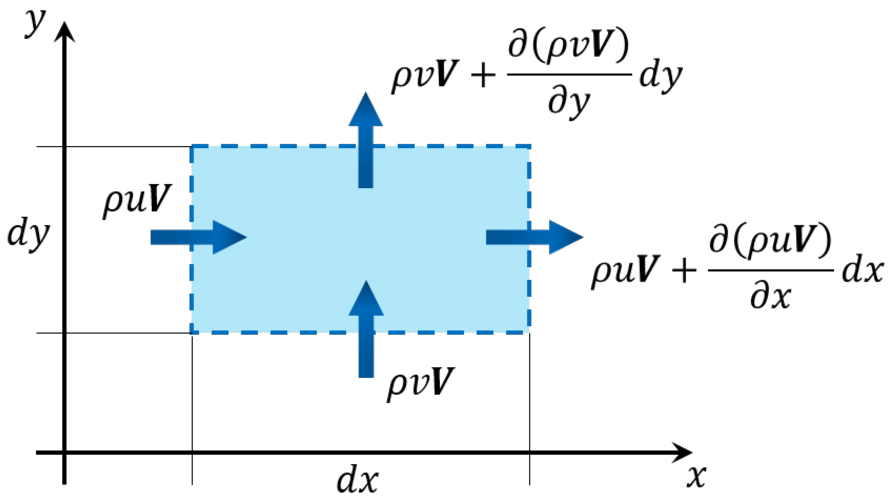


Figure 3: Control volume utilised to derive the equation governing the conservation of momentum.

3 Conservation of momentum

We consider the same two-dimensional control volume utilised to derive the continuity equation, see now Fig. 3. The conservation of momentum requires that the temporal variation of the momentum in the control volume plus the sum of the inflow/outflow fluxes through the control volume faces must be equal to the net force acting on the control volume. We denote as $\mathbf{Q}$ the momentum included in the control volume (units: $\mathrm{kg \cdot m/s}$), $\dot{\mathbf{Q}}_{\mathbf{in}}$ the momentum flow rate entering the control volume through the west and south faces (units: $\mathrm{kg \cdot m/s^2}$), and $\dot{\mathbf{Q}}_{\mathbf{out}}$ the momentum flow rate exiting the control volume through the north and east faces. Note that the momentum is a vectorial quantity and has as many components as the velocity field. Therefore, the conservation of momentum states that:

$$\frac{\partial \mathbf{Q}}{\partial t} + \dot{\mathbf{Q}}_{\mathbf{out}} - \dot{\mathbf{Q}}_{\mathbf{in}} = \sum_i \mathbf{F_i}, \tag{13}$$

where $\mathbf{F_i}$ represents the $i-th$ force acting on the control volume. Note that Eq. (13) has units $\mathrm{kg \cdot m/s^2}$, which coincides with newton, N. Let's now work out the left-hand side of the momentum equation. Following the same reasoning adopted for the continuity equation,

$$\mathbf{Q} = \int_{volume} \rho \mathbf{V}\, dx\, dy, \tag{14}$$

where the differential in the integral should be $dx\, dy\, 1$ to account for the control volume extension along z, but the 1 is dropped for simplicity of notation. Analogously to Eqs. (5)-(6):

$$\frac{\partial \mathbf{Q}}{\partial t} \approx \frac{\partial(\rho \mathbf{V})}{\partial t}\, dx\, dy. \tag{15}$$

To express $\dot{\mathbf{Q}}_{\mathbf{in}}$, we need to express the momentum flow rate through the west and south faces of the control volume, see the schematic in Fig. 3. Through the west face, there exists a mass flow rate $(\rho u)dy$ which carries a momentum flow rate $(\rho u)\mathbf{V}dy$; similarly, through the south face, there exists a mass flow rate $(\rho v)dx$ which carries a momentum flow rate $(\rho v)\mathbf{V}dx$, so that the total inflow of momentum is:

$$\dot{\mathbf{Q}}_{\mathbf{in}} = \rho u \mathbf{V}\, dy + \rho v \mathbf{V}\, dx. \tag{16}$$

The outgoing momentum flow rate requires knowledge of the momentum on the east and north faces, which can be expressed using the first-order Taylor expansion of the momentum as done previously for the mass flow rate, so that on the east face $(x + dx)$:

$$\left[\rho u\mathbf{V} + \frac{\partial(\rho u\mathbf{V})}{\partial x}dx\right] dy,$$

and on the north face $(y + dy)$:

$$\left[\rho v\mathbf{V} + \frac{\partial(\rho v\mathbf{V})}{\partial y}dy\right] dx.$$

This leads to:

$$\dot{\mathbf{Q}}_{\mathbf{out}} = \left[\rho u\mathbf{V} + \frac{\partial(\rho u\mathbf{V})}{\partial x}dx\right] dy + \left[\rho v\mathbf{V} + \frac{\partial(\rho v\mathbf{V})}{\partial y}dy\right] dx. \tag{17}$$

Introducing now Eqs. (15), (16), and (17) into Eq. (13):

$$\left[\frac{\partial(\rho\mathbf{V})}{\partial t} + \frac{\partial(\rho u\mathbf{V})}{\partial x} + \frac{\partial(\rho v\mathbf{V})}{\partial y}\right] dx\,dy = \sum_i \mathbf{F_i}, \tag{18}$$

where the element volume is actually $dx\,dy\,1$ (units: m^3) so that Eq. (18) is still in N units, as for Eq. (13). Note that the equation above is a vectorial equation and it has components along the x and y axes:

$$\left[\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y}\right] dx\,dy = \sum_i F_{i,x}, \tag{19a}$$

$$\left[\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y}\right] dx\,dy = \sum_i F_{i,y}. \tag{19b}$$

We can manipulate further the terms between square brackets in both Eqs. (19a) and (19b); let's consider Eqs. (19a) and apply the rule for developing the derivative of the products of functions:

$$\begin{aligned}\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} &= u\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + (\rho u)\frac{\partial u}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + (\rho v)\frac{\partial u}{\partial y} = \\ &= u\left[\cancel{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}}\right] + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho\left(\frac{\partial u}{\partial t} + \mathbf{V}\cdot\nabla u\right),\end{aligned} \tag{20}$$

where the term between square brackets on the second line is zero due to continuity. The operation $\mathbf{V}\cdot\nabla u$ indicates a scalar product between the vectors $\mathbf{V}$ and ∇u. It is left as an exercise to demonstrate that:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho\left(\frac{\partial v}{\partial t} + \mathbf{V}\cdot\nabla v\right). \tag{21}$$

4 Forces acting on a fluid in motion

We have sorted out the left-hand side of the momentum equation, let's now focus on the right-hand side, i.e. the forces acting on a fluid in motion. But which forces do act on a fluid in motion? We have:

- **Body forces**, which are proportional to the mass of the control volume, for instance the gravity force:

$$\mathbf{F_g} = M\mathbf{g} \approx \rho\mathbf{g}\, dx\, dy, \qquad \text{with} \quad \mathbf{g} = g_x\hat{\mathbf{i}} + g_y\hat{\mathbf{j}}, \tag{22}$$

  where we use the $\approx$ symbol because in reality we should perform an integral of $\rho\mathbf{g}$ in the control volume. Once more, note that the element volume is actually $dx\, dy\, 1$, so that the units of $\mathbf{F_g}$ are newton.

- **Surface forces**, which are the stresses acting along the surfaces of the control volume and are therefore proportional to the surface of the element. As we will see below, there are two kind of surface stresses: the (i) pressure and the (ii) viscous stresses, with the latter being zero for a fluid at rest.

5 Representation of the surface stresses

On each face of the control volume, there exist a stress normal to the surface and another one tangential to the surface, see the schematic in Fig. 4. The convention is to represent the normal stresses pointing outward the control volume, while the tangential stresses on opposite faces of the control volume must point on opposite directions. The stress acting on a generic face is denoted as $\sigma_{i,j}$, with i indicating that the stress applies on a face orthogonal to the $i-$axis, and j indicating that the stress is directed on the $j-$axis. Note that the units for σ are $\mathrm{N/m^2} \equiv \mathrm{Pa}$, i.e. a force per unit surface, where the surface is the area of the control volume face to which σ applies. From this convention, it follows that the normal and tangential stresses acting on the south face are, respectively, σ_{yy} and σ_{yx} (see Fig. 4), while on the west face we have σ_{xx} and σ_{xy}. Since the stresses depend on the space, the stresses on the east and north faces will be different from those on the other two faces. Again, we adopt the first-order Taylor expansion to express the stresses on the east and north faces as indicated in Fig. 4. At this point, we are able to express the total surface force acting along x (or y), $F_{s,x}$ (or $F_{s,y}$), by

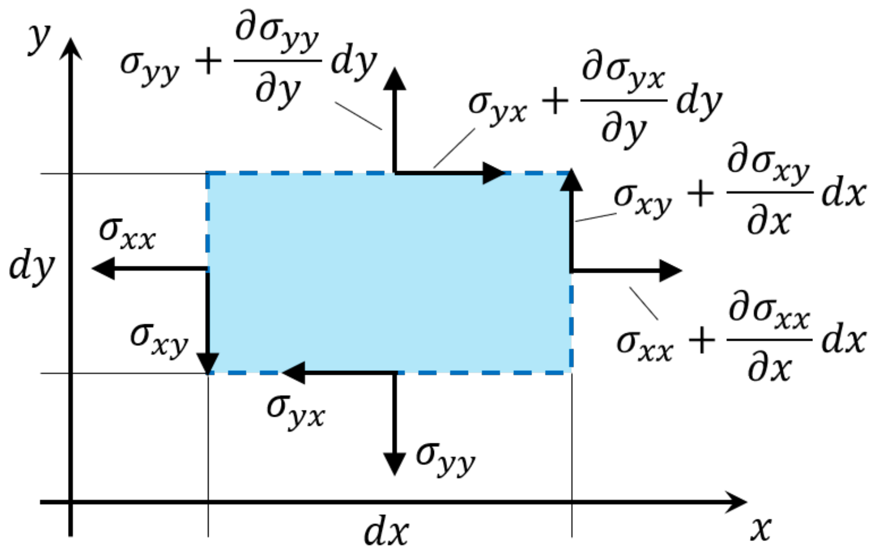


Figure 4: Representation of the surface stresses acting on a control volume in a fluid in motion.

summing the four horizontal (or vertical) contributions in Fig. 4, each multiplied by the area of the control volume face to which it applies (remember that each face has extension of 1 unit length along z). Starting from the west face, in the counter-clockwise direction:

$$\begin{aligned} F_{s,x} &= -\sigma_{xx}dy - \sigma_{yx}dx + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dy + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy\right)dx = \\ &= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)dx\,dy, \end{aligned} \tag{23}$$

and:

$$\begin{aligned} F_{s,y} &= -\sigma_{xy}dy - \sigma_{yy}dx + \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}dx\right)dy + \left(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}dy\right)dx = \\ &= \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)dx\,dy, \end{aligned} \tag{24}$$

where the element volume is actually $dx\,dy\,1$, so that the units of $F_{s,x}$ and $F_{s,y}$ are newton. It is interesting to note that it is not the stresses, but their gradients, that cause a net force on the control volume.

We can now replace Eqs. (22), (23), and (24) into Eq. (19) (with the manipulations in Eqs. (20) and (21)):

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dx\,dy = \sum_i F_{i,x} = F_{g,x} + F_{s,x} = \left[\rho g_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)\right]dx\,dy, \tag{25a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)dx\,dy = \sum_i F_{i,y} = F_{g,y} + F_{s,y} = \left[\rho g_y + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\right]dx\,dy, \tag{25b}$$

Note that the units are still newton (as the element has actual volume $dx\,dy\,1$). We can now drop the element volume as it appears at both right- and left-hand sides of the equations above, thus obtaining:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right), \tag{26a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right), \tag{26b}$$

which is a first form of the momentum equation. Note that the units of Eq. (26) are N/m^3.

As anticipated at the end of Section 4, the surface stresses are the sum of *pressure* plus *viscous stresses*, the latter due to the viscosity of the fluid and arising from the fluid motion (actually, from the velocity gradients, as we will see). The pressure is a normal stress, as it is always orthogonal to the control volume surface on which it acts, and it is a *compression* stress as it tends to shrink the control volume. Therefore, we can rewrite the normal stresses acting on the control volume surfaces using:

$$\sigma_{xx} = -p + \tau_{xx}, \tag{27a}$$

$$\sigma_{yy} = -p + \tau_{yy}, \tag{27b}$$

where p is the pressure (units: pascal, or N/m^2), negative because it compresses the control volume, while τ_{xx} and τ_{yy} (same units) indicate the shear stresses orthogonal to the control volume faces. The tangential stresses are composed of viscous stresses only, as pressure does not act tangentially to the control volume faces, and therefore:

$$\sigma_{xy} \equiv \tau_{xy}, \tag{28a}$$

$$\sigma_{yx} \equiv \tau_{yx}, \tag{28b}$$

which is merely a change of notation. Substituting Eqs. (27) and (28) in Eq. (26), we obtain:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \rho g_x-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}, \tag{29a}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = \rho g_y-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}, \tag{29b}$$

which represent the momentum equations (not yet the Navier-Stokes equations) and are valid for any fluid in any general two-dimensional motion. The units are the same as Eq. (26), i.e. N/m^3. In order to develop further the equation above, we need to make some assumptions about how the viscous stresses τ depend on the known quantitites.

Note that there are four forces appearing in Eq. (29), here with reference to Eq. (29a):

- $\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$ is the **inertial force** (here in the x direction) per unit volume of fluid.
- ρg_x is the **gravitational force** (here in the x direction) per unit volume of fluid.
- $-\frac{\partial p}{\partial x}$ is the **pressure force** (here in the x direction) per unit volume of fluid.
- $\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}$ is the force due to **viscous stress** (here in the x direction) per unit volume of fluid.

6 Newtonian fluids

In the module of Thermofluids 1, we have seen that (page 64 of TF1 notes) for a *newtonian* fluid in a straight, parallel and laminar flow over a fixed, horizontal wall, the shear stress τ in a direction parallel to the wall can be expressed as:

$$\tau = \mu\frac{du}{dy} \tag{30}$$

where u is the horizontal velocity (parallel to the wall), y indicates a direction perpendicular to the wall, and μ is the *dynamic viscosity* of the fluid (units: $kg/(m\cdot s)$, or $Pa\cdot s$). The dynamic viscosity is a property of the fluid and, in principle, it depends only on temperature and pressure, although in this module we will simply consider it constant. By extending Eq. (30) to the general case of a two-dimensional flow, the components of the viscous stress for a newtonian fluid can be expressed as:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{31a}$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{31b}$$

$$\tau_{xy} = \tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right). \tag{31c}$$

Equation (31) represents the constitutive equation for a newtonian fluid. We can now calculate the partial derivatives of the viscous stresses, that will be useful to express the momentum equation for a newtonian fluid (a bit lengthy, but it's just derivations),

$$\frac{\partial \tau_{xx}}{\partial x} = 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right), \tag{32a}$$

$$\frac{\partial \tau_{yx}}{\partial y} = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right), \tag{32b}$$

$$\frac{\partial \tau_{xy}}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right), \tag{32c}$$

$$\frac{\partial \tau_{yy}}{\partial y} = 2\mu\frac{\partial^2 v}{\partial y^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}\right), \tag{32d}$$

where the fluid viscosity has been assumed constant, so that:

$$\begin{aligned}\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} &= 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right) = \\ &= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \end{aligned} \tag{33a}$$

$$\begin{aligned}\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} &= \mu\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + 2\mu\frac{\partial^2 v}{\partial y^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}\right) = \\ &= \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \end{aligned} \tag{33b}$$

Equation (33) can be used to develop further the right-hand side of Eq. (29), which will be done in Section 7.

Incompressible flow

In the case of incompressible flow, for which Eq. (12) holds, the components of the viscous stresses for a newtonian fluid, Eq. (31), simplify as:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \tag{34a}$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \tag{34b}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{34c}$$

We can calculate the partial derivatives of the viscous stresses, that will be useful to express the momentum equations for a newtonian fluid in incompressible flow,

$$\frac{\partial \tau_{xx}}{\partial x} = 2\mu\frac{\partial^2 u}{\partial x^2}, \tag{35a}$$

$$\frac{\partial \tau_{yx}}{\partial y} = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right), \tag{35b}$$

$$\frac{\partial \tau_{xy}}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right), \tag{35c}$$

$$\frac{\partial \tau_{yy}}{\partial y} = 2\mu\frac{\partial^2 v}{\partial y^2}, \tag{35d}$$

so that:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x} \right) = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \tag{36a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = \mu \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right) + 2\mu \frac{\partial^2 v}{\partial y^2} = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \tag{36b}$$

where the two last terms at the right-hand sides are zero because of continuity.

7 Navier-Stokes equations

Let's start with the generic case of a newtonian fluid and compressible flow, for which the viscous stresses can be expressed with Eq. (33). Replacing Eq. (33) into Eq. (29), we obtain the **Navier-Stokes equations** for a newtonian fluid in two dimensions:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{1}{3} \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \tag{37a}$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{1}{3} \mu \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \tag{37b}$$

The units are N/m^3. Now it is evident that, for a newtonian fluid, the viscous stresses arise from the velocity gradients; if the velocity is constant, the viscous stresses are zero.

Incompressible flow

Replacing Eqs. (36) into Eq. (29), we obtain the Navier-Stokes equations for a newtonian fluid in incompressible flow in two dimensions:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{38a}$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \tag{38b}$$

Again, note that there are four forces appearing in Eq. (38), here with reference to Eq. (38a):

- $\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)$ is the **inertial force** (here in the x direction) per unit volume of fluid.
- ρg_x is the **gravitational force** (here in the x direction) per unit volume of fluid.
- $-\frac{\partial p}{\partial x}$ is the **pressure force** (here in the x direction) per unit volume of fluid.
- $\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ is the force due to **viscous stress** (here in the x direction) per unit volume of fluid.

8 General form of mass and momentum equations

This section provides a summary of the equations seen in the previous sections.

Mass/momentum equations for a newtonian fluid in 2D

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{39a}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{39b}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{39c}$$

Mass/momentum equations for a newtonian fluid in incompressible flow in 2D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{40a}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{40b}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{40c}$$

Mass/momentum equations for a newtonian fluid in 3D

These were not derived before, but the derivation follows naturally by extending the 2D case to 3D.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{41a}$$

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= \rho g_x - \frac{\partial p}{\partial x} + \\ &+ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),\end{aligned} \tag{41b}$$

$$\begin{aligned}\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= \rho g_y - \frac{\partial p}{\partial y} + \\ &+ \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),\end{aligned} \tag{41c}$$

$$\begin{aligned}\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= \rho g_z - \frac{\partial p}{\partial z} + \\ &+ \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).\end{aligned} \tag{41d}$$

Mass/momentum equations for a newtonian fluid in incompressible flow in 3D

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{42a}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{42b}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{42c}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{42d}$$

Mass/momentum equations in compact notation

Finally, we show that we can recompact the equations using the del operator, ∇. This is very useful, because it makes very clear the similarities among the various terms of the equations, and among their components along the coordinate axes. Bear in mind the definition of the del operator, here for 3D Cartesian orthogonal coordinates:

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}, \tag{43}$$

therefore we can express the divergence of a vector field $\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ as:

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}, \tag{44}$$

and the laplacian of a scalar field b as:

$$\nabla^2 b = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2}. \tag{45}$$

Using these operators, the mass conservation equation takes the same form in 2D and 3D:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{46}$$

and, if the flow is incompressible:

$$\nabla \cdot \mathbf{V} = 0. \tag{47}$$

The Navier-Stokes equations in the general case of a compressible flow can be rewritten as:

$$\rho\left(\frac{\partial u}{\partial t} + \mathbf{V}\cdot\nabla u\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\nabla^2 u + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\nabla\cdot\mathbf{V}\right), \tag{48a}$$

$$\rho\left(\frac{\partial v}{\partial t} + \mathbf{V}\cdot\nabla v\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\nabla^2 v + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\nabla\cdot\mathbf{V}\right), \tag{48b}$$

$$\rho\left(\frac{\partial w}{\partial t} + \mathbf{V}\cdot\nabla w\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\nabla^2 w + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\nabla\cdot\mathbf{V}\right), \tag{48c}$$

and, in vectorial form:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{V} + \frac{1}{3}\mu\nabla\left(\nabla\cdot\mathbf{V}\right). \tag{49}$$

For an incompressible flow,

$$\rho\left(\frac{\partial u}{\partial t}+\mathbf{V}\cdot\nabla u\right) = \rho g_x - \frac{\partial p}{\partial x}+\mu\nabla^2 u, \tag{50a}$$

$$\rho\left(\frac{\partial v}{\partial t}+\mathbf{V}\cdot\nabla v\right) = \rho g_y - \frac{\partial p}{\partial y}+\mu\nabla^2 v, \tag{50b}$$

$$\rho\left(\frac{\partial w}{\partial t}+\mathbf{V}\cdot\nabla w\right) = \rho g_z - \frac{\partial p}{\partial z}+\mu\nabla^2 w, \tag{50c}$$

and, in vectorial form:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla \mathbf{V}\right) = \rho\mathbf{g} - \nabla p+\mu\nabla^2 \mathbf{V}. \tag{51}$$

Therefore, with an eye on Eq. (50), the similarity among the x, y, z components of the Navier-Stokes equations is evident and this helps to memorise them, all the equations include:

- The temporal term of the inertial force, with the operator $\frac{\partial}{\partial t}$ applied to the velocity component (u, v, w) for which the equation is being written.
- The convective term of the inertial force, with the operator $(\mathbf{V}\cdot\nabla)$ applied to the velocity component.
- The x, y or z component of the external body force, here the gravitational force.
- The pressure force, expressed by the partial derivative of the pressure in the direction along which the equation refers to.
- The viscous force, with the operator ∇^2 applied to the velocity component.

9 Solution of the Navier-Stokes equations

The mass and momentum equations represent a system of one scalar equation (mass conservation) and one vectorial equation (momentum conservation). In the most general case (compressible flow), there are two scalar unknowns, the pressure p and density ρ, and one vectorial unknown, the velocity $\mathbf{V}$. Therefore, one additional equation is needed to close the problem. This is usually an equation of state that relates ρ to pressure and temperature, $\rho = \rho(p, T)$, such as the ideal gas law. However, note that if the temperature varies in space and time, then we need another conservation equation for the energy in order to have the same number of unknowns and equations. In the case of incompressible flow, the density is a known constant and therefore the only unknowns are p and $\mathbf{V}$, so that in this case the mass and momentum equations are sufficient to describe the problem.

Boundary conditions

The mass and momentum equations constitute a system of partial differential equations with space and time as independent variables, and therefore we need initial (at $t = 0$) and boundary conditions in order to solve them. Figure 5 shows a representative example of the flow of a fluid between two parallel plates, where the fluid enters from the left boundary and exits through the right boundary. Typical initial and boundary conditions in engineering problems are:

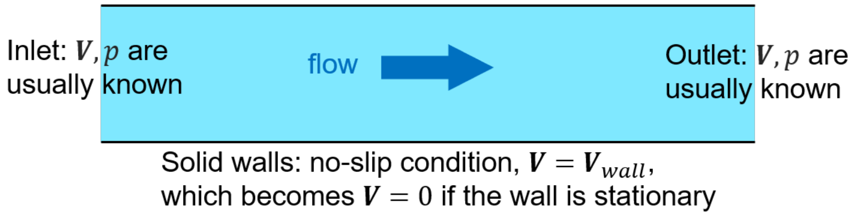


Figure 5: Schematic of the boundary conditions that apply to velocity and pressure fields for flow between two parallel plates; the flow is from left to right.

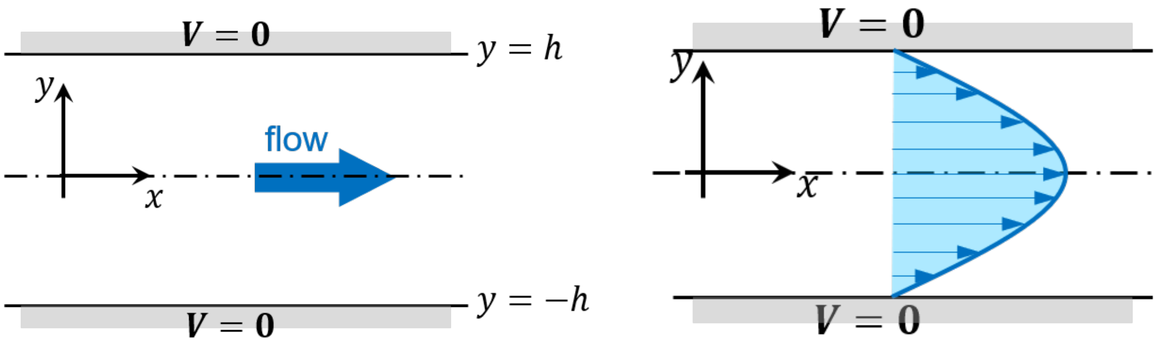
- Initial conditions ($t = 0$): Velocity and pressure must be known functions of x, y, z. Note that if the flow is steady, so that all the terms $\partial/\partial t$ are zero in the flow equations, no initial conditions are necessary.
- *Inlet/outlet*: Velocity and pressure are usually known.
- *Walls*: Usual boundary conditions are *no-through* and *no-slip*. The no-through boundary condition requires that no fluid flows through the wall, and therefore the component of the velocity of the fluid normal to the wall (say $V_\perp$) must be equal to the normal velocity of the wall (say $V_{w,\perp}$), $V_\perp = V_{w,\perp}$; if the wall is stationary, it follows that $V_\perp = 0$. The no-slip boundary condition requires that, because of viscosity, the fluid element next to the wall must adhere to it, and therefore the component of the velocity of the fluid parallel to the wall (say $V_\parallel$) must be equal to the parallel velocity of the wall (say $V_{w,\parallel}$), $V_\parallel = V_{w,\parallel}$; if the wall is stationary, it follows that $V_\parallel = 0$.

10 Analytical solution of the Navier-Stokes equations

The Navier-Stokes equations can be solved in closed form only for a very few simple configurations, below we provide two of such examples.

Laminar flow between two parallel plates

We consider the incompressible, steady state flow of a newtonian fluid between two infinitely extended (stationary) parallel plates, see the schematic in Fig. 6(a). The flow is assumed laminar, and therefore the Reynolds number of the flow, $\mathrm{Re} = \rho u_{mean} \ell/\mu$, with u_{mean} being the average fluid velocity on the cross-section and $\ell = 4h$ a characteristic length of the cross-section (twice the distance between the parallel plates), is below 2300. Because the plates are infinitely extended in the depth (z) direction, the flow can be regarded as two-dimensional (with $w = 0$). Additionally, we assume that the flow is driven by a constant streamwise pressure gradient $\partial p/\partial x$, and that the flow is essentially axial, with $u = u(x, y) \neq 0$, but $v = 0$; these assumptions have been observed to be valid when considering a region of the channel far from the entrance. The gravitational force can be neglected. The scope of the exercise is to derive the velocity profile on the channel cross-section.



(a) (b)

Figure 6: (a) Schematic of the channel geometry and reference frame for flow between two parallel plates and (b) resulting parabolic velocity profile.

Let's start be recalling the continuity equation in the case of an incompressible flow (2D):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{52}$$

but $v = 0$ as stated in the text of the exercise, and thus it follows from the equation above that:

$$\frac{\partial u}{\partial x} = 0, \tag{53}$$

and therefore u is a function of y only, $u = u(y)$. From the Navier-Stokes equations for a newtonian fluid in incompressible flow, the $x-$momentum equation writes as:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{54}$$

where the first term at the left-hand side is zero because the flow is steady-state, the second term is zero because of continuity, the third term is zero because $v = 0$, and the second term at the right-hand side is zero because $u \neq u(x)$, so that the equation above simplifies as:

$$\mu\frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x}. \tag{55}$$

Note that the partial derivative of u with respect to y becomes an ordinary derivative because u depends only on y. The $y-$momentum equation writes as:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{56}$$

however $v = 0$ and therefore all the terms but the pressure gradient are zero:

$$\frac{\partial p}{\partial y} = 0, \tag{57}$$

from which it follows that p is a function of x only, $p = p(x)$, and thus it is constant on the cross-section (as it is not a function of y). This allows us to rewrite the partial derivative of the pressure in Eq. (55) as an ordinary derivative:

$$\mu\frac{d^2 u}{dy^2} = \frac{dp}{dx}, \tag{58}$$

where we know from the text of the exercise that dp/dx is constant. Equation (58) is a second-order ordinary differential equation that can be easily solved with two boundary conditions. By applying no-slip conditions at the channel walls, we know that:

$$u(y = \pm h) = 0. \tag{59}$$

Integrating twice Eq. (58) with respect to y we obtain the velocity profile on the cross-section:

$$u(y) = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2, \tag{60}$$

and, imposing the boundary conditions at the walls:

$$u(y = +h) \quad = \quad 0 \Rightarrow \frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2} + C_1 h + C_2 = 0, \tag{61a}$$

$$u(y = -h) \quad = \quad 0 \Rightarrow \frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2} - C_1 h + C_2 = 0, \tag{61b}$$

which can be manipulated to extract the integration constants:

$$C_1 = 0, \quad C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}, \tag{62}$$

so that the velocity profile is now completely known:

$$u(y) = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right). \tag{63}$$

The velocity profile is parabolic, with $u = 0$ at the walls owing to the no-slip condition, and maximum velocity at the centre ($y = 0$):

$$u_{max} = u(y = 0) = -\frac{dp}{dx}\frac{h^2}{2\mu}, \tag{64}$$

see the sketch in Fig. 6(b). Note that the pressure decreases along x due to the wall shear, and hence $dp/dx < 0$, such that $u > 0$.

Now that we have the velocity profile, we can extract some useful quantities, for example the volumetric flow rate $\dot{Q}$:

$$\dot{Q} = \int_{-h}^{+h} u(y)dy = -\frac{dp}{dx}\frac{2h^3}{3\mu}, \tag{65}$$

and the average flow velocity u_{mean}:

$$u_{mean} = \frac{1}{2h}\int_{-h}^{+h} u(y)dy = \frac{\dot{Q}}{2h} = -\frac{dp}{dx}\frac{h^2}{3\mu}, \tag{66}$$

where $2h$ is the distance between the two plates. It follows that:

$$u_{max} = \frac{3}{2}u_{mean}, \tag{67}$$

and thus we can rewrite the velocity profile as:

$$u(y) = \frac{3}{2}u_{mean}\left(1 - \frac{y^2}{h^2}\right). \tag{68}$$

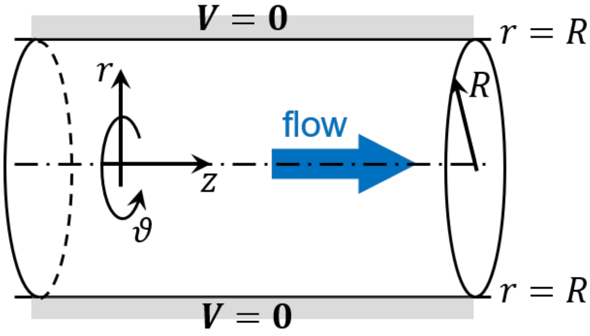


Figure 7: Schematic of the channel geometry and reference frame for flow in a circular pipe.

Note that the units of the volumetric flow rate $\dot{Q}$ should be m^3/s, although from Eq. (65) we find m^2/s because we are not considering the extension of the channel along z. From the engineering point of view, it is useful to relate the total pressure drop for a channel of length L to the average flow rate. Separating the differentials in Eq. (65)

$$dp = -\frac{3\mu\dot{Q}}{2h^3}dx \Rightarrow \Delta p = -\int_0^L dp = -\int_0^L -\frac{3\mu\dot{Q}}{2h^3}dx = \frac{3\mu\dot{Q}L}{2h^3}, \tag{69}$$

and, in terms of mean fluid velocity:

$$\Delta p = \frac{3\mu u_{mean} L}{h^2}. \tag{70}$$

Always double-check the units to make sure that calculations are correct, the pressure drop Δp is always in pascal.

Laminar flow in a circular pipe

Now we repeat the derivation of the velocity profile, but for a circular pipe geometry, see the schematic in Fig. 7. We consider the flow laminar, incompressible and steady-state, the fluid newtonian. It is convenient to use a cylindrical reference frame of coordinates r, θ, z, with velocity components u_r, u_θ, u_z. We assume that the flow is driven by a constant streamwise pressure gradient $\partial p/\partial z$, and that the flow is essentially axial, with $u_z \neq 0$, but $u_r = u_\theta = 0$; we also assume axial symmetry, i.e. $\partial/\partial\theta = 0$, so that velocity and pressure are independent of θ. These assumptions have been observed to be valid when considering a region of the channel far from the entrance. The gravitational force can be neglected. The scope of the exercise is to derive the velocity profile on the channel cross-section.

We need now to write down the mass and momentum equations for cylindrical coordinates. Although the vectorial form of the mass and momentum equations for incompressible flow and newtonian fluid, Eqs. (47) and (51), hold independently of the coordinate system, the del operator in cylindrical coordinates writes differently:

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{\mathbf{z}}, \tag{71}$$

and hence also the divergence operator, laplacian, etc., write differently. These are not reported here, but the interested reader can take a look at the following Wikipedia page. The mass and momentum equations for incompressible flow and newtonian fluid in cylindrical coordinates, for axial symmetry ($\partial/\partial\theta = 0$ and $u_\theta = 0$), write as follows:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0, \tag{72a}$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right], \tag{72b}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right]. \tag{72c}$$

From the continuity equation, Eq. (72a), we can eliminate the first term as the exercise suggests that $u_r = 0$, therefore:

$$\frac{\partial u_z}{\partial z} = 0, \tag{73}$$

and therefore u_z is a function of r only, $u_z = u_z(r)$. From the $r-$momentum equation, Eq. (72b), all the terms are zero because $u_r = 0$ except for the pressure gradient, and thus it follows that:

$$\frac{\partial p}{\partial r} = 0, \tag{74}$$

pressure is a function of z only, $p = p(z)$, and thus it is constant on the cross-section (as it is not a function of r). From the $z-$momentum equation, Eq. (72c), the first term at the left-hand side is zero because the flow is steady, the second term is zero because $u_r = 0$, and the third term is zero because of continuity; at the right-hand side, the partial derivative of the pressure becomes an ordinary derivative because of Eq. (74), and the last term is zero because $u_z \neq u_z(z)$. As such, Eq. (72c) simplifies as:

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{dp}{dz}, \tag{75}$$

where we know from the text of the exercise that dp/dz is constant. Note that the derivative of u_z with respect to r has become an ordinary derivative as u_z depends only on r. Equation (75) is a second-order ordinary differential equation; multiplying the two sides by r/μ and integrating once with respect to r, we obtain:

$$r\frac{du_z}{dr} = \frac{dp}{dz}\frac{r^2}{2\mu} + C_1, \tag{76}$$

dividing everything by r and integrating a second time, we obtain:

$$u_z(r) = \frac{dp}{dz}\frac{r^2}{4\mu} + C_1 \ln r + C_2. \tag{77}$$

Now we need to impose the boundary conditions to obtain C_1 and C_2. The no-slip condition at the wall yields:

$$u_z(r = R) = 0 \Rightarrow \frac{dp}{dz}\frac{R^2}{4\mu} + C_1 \ln R + C_2 = 0, \tag{78}$$

where R is the radius of the pipe. The second boundary condition is obtained by requiring that the profile is finite at $r = 0$, where the logarithm would make the solution diverge to infinite, so that C_1 must be zero. This yields:

$$C_1 = 0, \quad C_2 = -\frac{dp}{dz}\frac{R^2}{4\mu}, \tag{79}$$

from which the velocity profile is fully determined:

$$u_z(r) = -\frac{dp}{dz}\frac{R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right), \tag{80}$$

which is parabolic in r and is known as the **Poiseuille velocity profile**. Note that $u_z > 0$ because $dp/dz < 0$, as explained in the previous exercise.

From the known velocity profile, we can now derive some useful quantities, for instance the maximum velocity at the pipe centre ($r = 0$):

$$u_{z,max} = -\frac{dp}{dz}\frac{R^2}{4\mu}, \tag{81}$$

the volumetric flow rate in the pipe (units: m^3/s):

$$\dot{Q} = \int_0^R u_z(r) 2\pi r \, dr = -\frac{dp}{dz}\frac{\pi R^4}{8\mu}, \tag{82}$$

and the average flow velocity in the pipe:

$$u_{z,mean} = \frac{1}{\pi R^2}\int_0^R u_z(r) 2\pi r \, dr = \frac{\dot{Q}}{\pi R^2} = -\frac{dp}{dz}\frac{R^2}{8\mu}, \tag{83}$$

from which it follows that $u_{z,max} = 2u_{z,mean}$. As done for the previous exercise, it is useful to relate the pressure drop in a pipe of length L to the volumetric flow rate. Separating the differentials in Eq. (82) and integrating along z, from 0 to L:

$$\Delta p = \frac{8\mu\dot{Q}L}{\pi R^4}, \tag{84}$$

note that the pressure drop increases dramatically as the radius of the pipe is reduced, due to the $\Delta p \sim 1/R^4$ dependence.