For all questions assume σ_y = 250 MPa for steel.

1. Using the Tresca yield criterion, determine the maximum allowable pure torque that can be applied to a 50mm solid circular steel shaft to avoid yield.
   [Ans.: 3068 Nm]

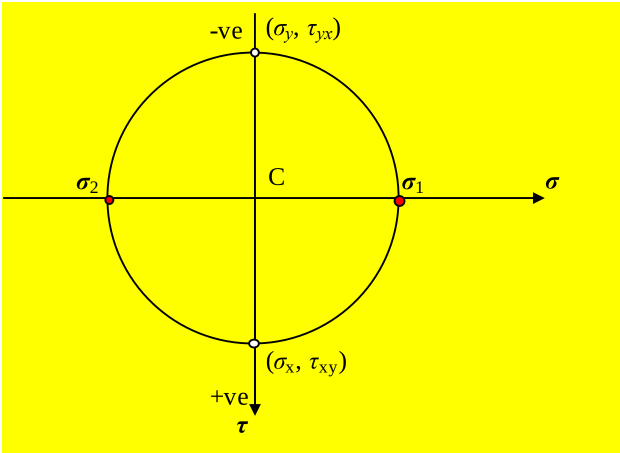
   Assuming a plane stress element on the outer surface of the shaft.
   For Tresca, $\tau_{max} = \frac{\sigma_y}{2}$ at yield, therefore $\tau_{max} = \frac{250}{2}$ = 125 MPa

   Recalling $\tau = \frac{Tr}{J}$, the value of maximum torque can be calculated by:
   $T = \frac{\tau J}{r}$, where $J = \frac{\pi d^4}{32}$

   so, $T = \frac{\tau_{max} \pi d^4}{32 r} = \frac{125\times10^6\times\pi\times(50\times10^{-3})^4}{32\times25\times10^{-3}}$ = ***<u>3068 Nm</u>***

2. Using the von Mises yield criterion, determine the maximum allowable pure torque that can be applied to a 50mm solid circular steel shaft to avoid yield.
   [Ans.: 3534 Nm]

   Assuming a plane stress element on the outer surface of the shaft.
   For von Mises, $\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$ for a plane stress case.
   In the case of pure torsion, considering Mohr's circle, which is centred on the origin:



   the principal stresses σ_1 and σ_2 are the same magnitude. Therefore, we can call this magnitude *k* and express the problem as $3k^2 = \sigma_y^2$ and the magnitude can be determined by $k = \frac{\sigma_y}{\sqrt{3}} = \frac{250}{\sqrt{3}}$ = 144 MPa.

The magnitudes of the principal stresses in this case also correspond to the maximum allowable shear stress τ_{max}, therefore:

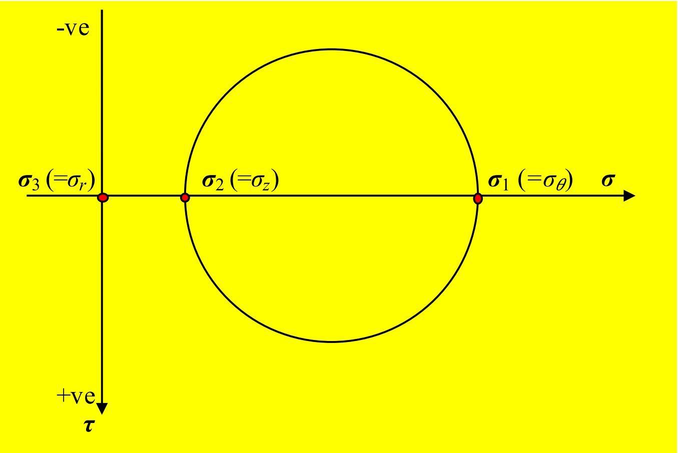
$T = \frac{\tau_{max}\pi d^4}{32r} = \frac{144\times10^6\times\pi\times(50\times10^{-3})^4}{32\times25\times10^{-3}}$ = ***3534 Nm***

3. Calculate the pressure to cause yielding in a steel cylinder, 80 mm diameter, 1 mm thick using:
   i) the Tresca yield criterion,
   ii) the von Mises yield criterion.
   The cylinder is closed at each end; end effects should be neglected.
   [Ans.: i) 6.25 MPa; ii) 7.22 MPa]

Given *D* = 80 mm, *R* = 40mm and *t* = 1 mm, $\frac{t}{R} < \frac{1}{10}$ therefore the cylinder is thin-walled.

The hoop stress in a thin-walled cylinder is given by $\sigma_\theta = \frac{pR}{t}$ and the axial stress is given by $\sigma_z = \frac{pR}{2T}$ and we can assume that the radial stress $\sigma_r = 0$.
Applying to this case to determine the pressure $\sigma_\theta = \frac{pR}{t} = 40p$, $\sigma_z = \frac{pR}{2T} = 20p$, $\sigma_r = 0$

In this case, both σ_θ and σ_z will be positive and aligned with the principal axes as there is no applied shear stress and the Mohr's circle will (conceptually) be:



i) Note that in this case it is important that we consider the radial stress as the third principal stress where $\sigma_1 > \sigma_2 > \sigma_3$

For Tresca, $\sigma_1 - \sigma_3 = \sigma_y$ at yield, therefore $\sigma_1 - \sigma_3$ = 250 MPa and as $\sigma_3 = \sigma_r = 0$, this means that at yield $\sigma_1 = 40p = \sigma_y$ and $p = \frac{250}{40}$ = ***6.25 MPa***

ii) for von Mises, $\sigma_y = \frac{1}{\sqrt{2}}((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2)^{\frac{1}{2}}$ at yield, therefore in this case, recalling that $\sigma_1 = \sigma_\theta = 40p, \sigma_2 = \sigma_z = 20p, \sigma_3 = 0$:

$$\sigma_y = \frac{1}{\sqrt{2}}((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}((40p - 20p)^2 + (20p - 0)^2 + (0 - 40p)^2)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(2400p^2)^{\frac{1}{2}}$$

or $250 = 34.64p$ or $p = \frac{250}{34.64}$ = ***7.22 MPa***

Directly using the plane stress version of the von Mises criterion ($\sigma_y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$):

$\sigma_y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 1600p^2 - 800p^2 + 400p^2 = 1200p^2$
or $250 = 34.64p$ *or* p = **7.22 MPa**

4. Recalculate the pressure values in question 3 if there is an additional constant axial tensile stress of 150 MPa in the cylinder using:
   i) the Tresca yield criterion,
   ii) the von Mises yield criterion.
   [Ans.: i) 5 MPa; ii) 5.77 MPa]

In this case, $\sigma_r = 0$ and $\sigma_\theta = 40p$ again, however σ_z will be $\sigma_z = \frac{pR}{2t} + 150 = 20p + 150$ so in this case, either σ_θ or σ_z could now be the maximum principal stress, σ_1 we can't be sure.

As the Tresca criterion requires that $\sigma_1 > \sigma_2 > \sigma_3$, we can make an initial assumption that σ_θ is σ_1 and try the Tresca criterion:
Again $\sigma_1 - \sigma_3 = \sigma_y$, this would mean again that $\sigma_1 = 40p = \sigma_y = 250$ and $p = \frac{250}{40}$ = 6.25 MPa. If we put this into the expression for $\sigma_z = 20p + 150 = 275$ MPa, which is greater than the value of σ_θ in this case, so this assumption must be wrong.

So, the maximum principal stress, σ_1 in this case must be σ_z and this now means that $\sigma_1 = \sigma_z = 20p + 150, \sigma_2 = \sigma_\theta = 40p, \sigma_3 = 0$.

As $\sigma_1 - \sigma_3$ = 250 MPa, this means that $20p + 150 = 250$ or
$p = \frac{250-150}{20} = \frac{150}{20}$ = ***5 MPa***

A $\sigma_3 = 0$, we can directly use the plane stress version of the von Mises criterion ($\sigma_y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$):

$$\sigma_y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = (20p + 150)^2 - 40p(20p + 150) + 40p^2$$
$$= 400p^2 + 6000p + 150^2 - 800p^2 - 6000p + 1600p^2$$
$$= 1200p^2 + 150^2 = 250^2$$

$p^2 = \frac{40000}{1200} = 33.3$, which implies p = ***5.77 MPa***

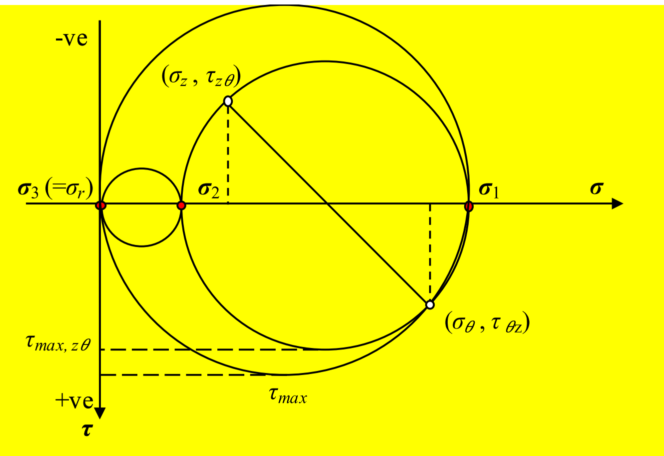
5. What additional torque can be applied about the axis of the cylinder in question 3 if yielding is to occur with an internal pressure of 4.0 MPa using:
   i) the Tresca yield criterion,
   ii) the von Mises yield criterion.
   [Ans.: i) 1146 Nm ii) 1162 Nm]

It is important here to distinguish between the overall maximum shear stress for the stress state in this case, τ_{max}, the maximum in plane shear stress for the z-θ plane that were interested in, $\tau_{max,z\theta}$, and the shear stress resulting from the torque in this plane $\tau_{z\theta}$.

For a plane stress element on the surface, again $\sigma_r = 0$, $\sigma_\theta = 40p = 160$ MPa and $\sigma_z = 20p = 80$ MPa and now $\tau_{z\theta} = \frac{TR}{J}$. We can also assume that the other shear stresses $\tau_{rz} = \tau_{r\theta} = 0$. Also, $J = \frac{\pi}{32}(D_o^4 - D_i^4)$

As before, $\sigma_r = 0$ will be one of the principal stresses, but we cannot be sure which one, there are two possibilities for how the 3D Mohr's circle may look in this case (conceptually):

Possibility 1: I this case *T* is not big enough for the minimum principal stress on the circle representing the *z*-θ plane to be less than σ_r:

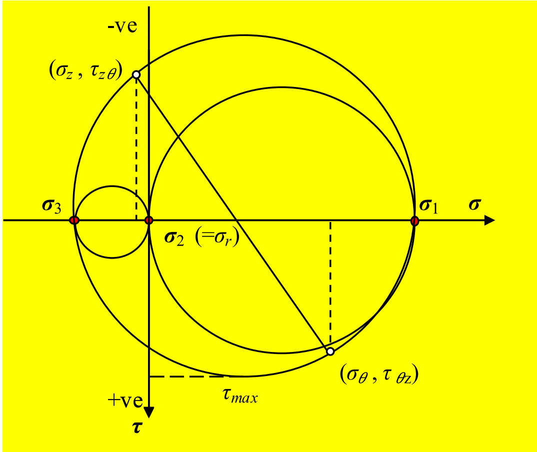


In this case, $\tau_{max} = \frac{\frac{\sigma_z + \sigma_\theta}{2} + \tau_{max,z\theta}}{2}$

For Tresca, yield occurs when $\tau_{max} = \frac{\sigma_y}{2} = 125$ MPa

Therefore, when $120 + \tau_{max,z\theta} = 250$ MPa or $\tau_{max,z\theta} = 130$ MPa which is not a feasible solution as the minimum principal stress in this plane would then be *C* - *R* = 120 - 130 = -10 MPa, which is less than $\sigma_r = 0$

Possibility 2: *T* is large enough for the minimum principal stress in the *z*-*θ* plane to be less than σ_r:



therefore for Tresca, $\tau_{max} = \tau_{max,z\theta} = 125$ MPa and $\tau_{max} = \tau_{max,z\theta} =$

$$\sqrt{\left(\frac{\sigma_\theta - \sigma_z}{2}\right)^2 + \tau_{z\theta}^2} = \sqrt{\left(\frac{80}{2}\right)^2 + \left(\frac{RT}{\frac{\pi}{32}(D_o^4 - D_i^4)}\right)^2} = \sqrt{40^2 + \left(\frac{32RT}{\pi(D_o^4 - D_i^4)}\right)^2} = 125 \text{ MPa},$$

which gives *T* = ***1146 Nm***

For von Mises, in plane stress ($\sigma_r = 0$) we can let $\sigma_1 = 120 + \sqrt{\tau^2 + 40^2}$, $\sigma_2 = 120 - \sqrt{\tau^2 + 40^2}$ and $\sigma_3 = 0$ and use $\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$

As $\sigma_1 = C + R$ and $\sigma_2 = C - R$, this means that $\sigma_y^2 = C^2 + 3R^2$ or $250^2 = 120^2 + 3\tau_{max}^2$, rearranging gives $\tau_{max}^2 = \frac{250^2 - 120^2}{3} = 126.6$ MPa

This means that in this case, $\sqrt{40^2 + \left(\frac{32RT}{\pi(D_o^4 - D_i^4)}\right)^2} = 126.6$ MPa, which gives

T = ***1162 Nm***