10 Thick-Walled Cylinders

Learning Summary

1. Appreciate the difference between the stress analysis of thin and thick cylinders (knowledge)
2. Be able to derive the equilibrium equations for a solid body (thick cylinder) (comprehension);
3. Understand the derivation of Lame's equations comprehension);
4. Determine the stresses caused by shrink fitting a cylinder onto another (application);
5. Be able to include 'inertia' effects into the thick cylinder equations to calculate the stresses in a rotating disc (application).

10.1 Introduction

Thick cylinders differ from thin cylinders in that the variation of stress through the wall thickness is significant when subjected to internal and/or external pressure whereas for thin cylinders, the variation of stress is negligible. Figure 10.1 presents the cases of thick cylinders with closed ends and with pistons. For closed-ended, internally pressured cylinders the axial force on the inside of the end closures produces a distribution of axial stress in the cylinder while for cylinders with pistons the resultant axial force in the cylinder and hence the axial stress also are zero.

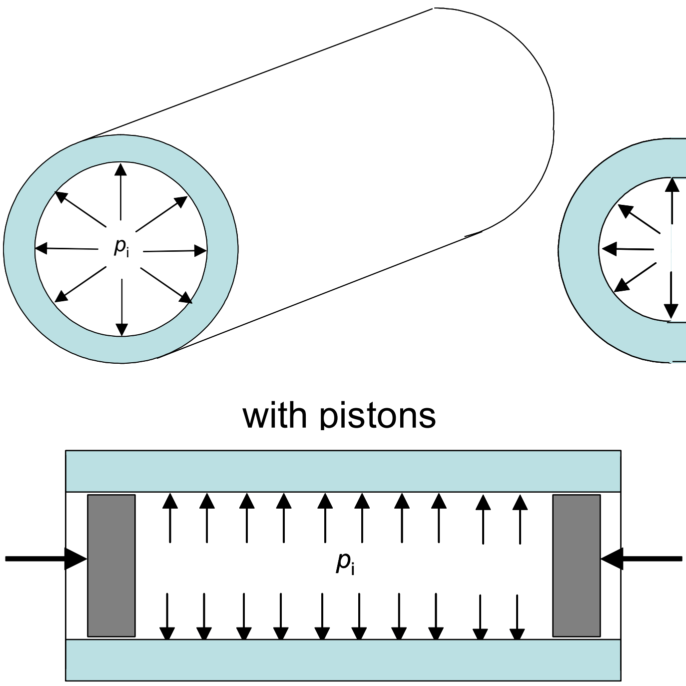


Figure 10.1: Thick cylinders subjected to internal pressure

10.2 Analysis of thin cylinders (recap)

For an internally pressurised thin cylinder situation, it is reasonable to assume that the variations of the stresses through the wall thickness are negligible resulting in the problem being *statically determinate*, i.e. expressions for the stresses can be obtained by consideration of equilibrium alone, as shown in Figure 10.2 and described below.

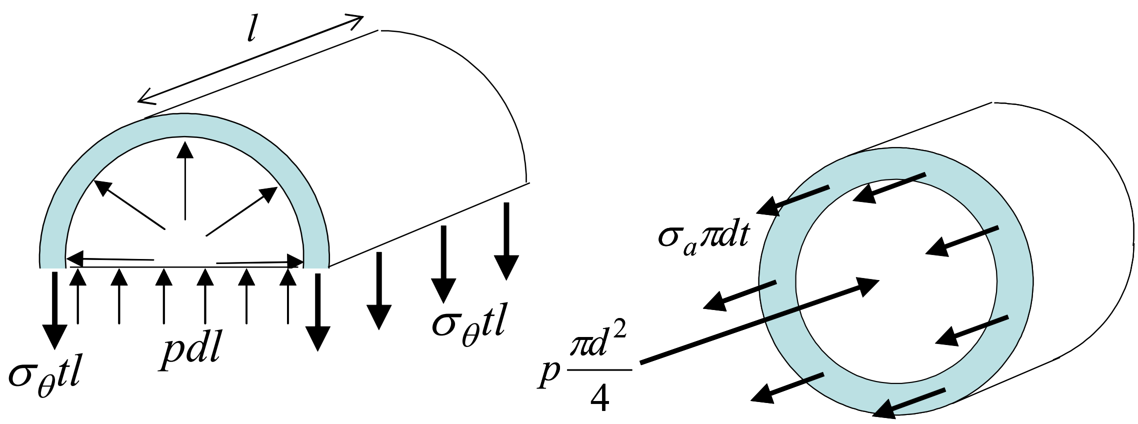


Figure 10.2: Hoop and axial stress in thin cylinders

$$2\sigma_\theta tl = pdl$$

$$\sigma_\theta = \frac{pd}{2t} = \frac{pR}{t} \tag{30}$$

$$\sigma_a \pi dt = p\frac{\pi d^2}{4}$$

$$\sigma_a = \frac{pd}{4t} = \frac{pR}{2t} \tag{31}$$

10.3 Analysis of thick cylinders

Thick cylinder problems are *statically indeterminate*. Therefore, in order to obtain a solution it is necessary to consider equilibrium, compatibility and the material behaviour (stress-strain relationship).

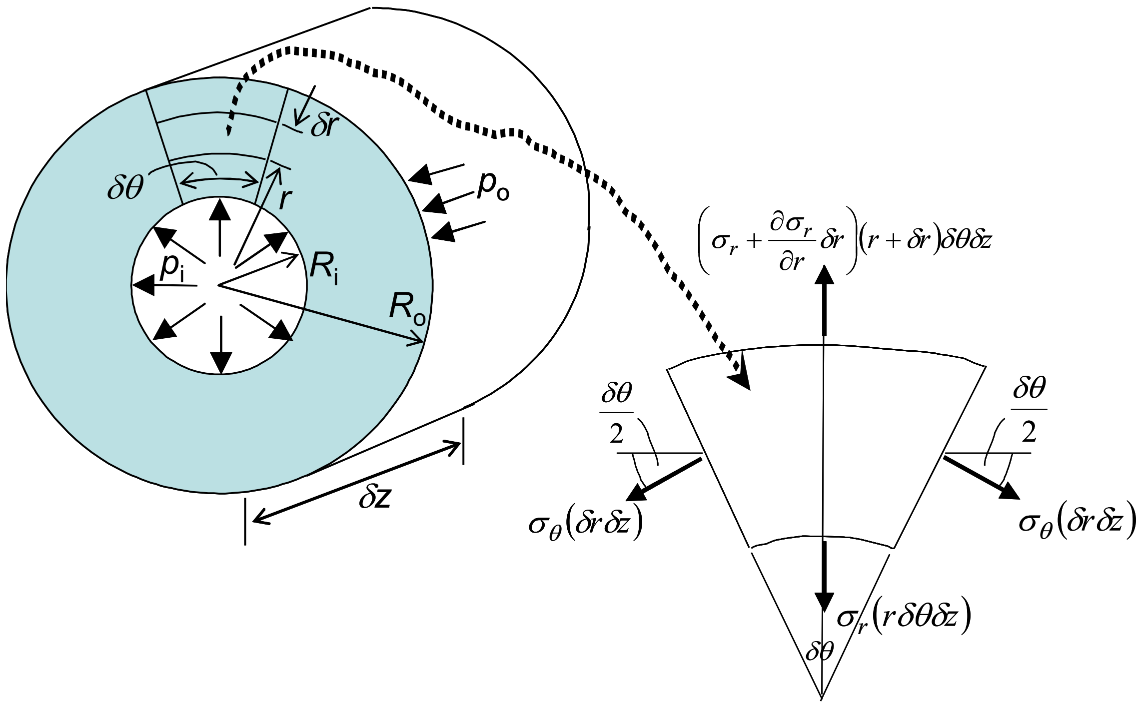
Assumptions

(i) Plane transverse sections remain plane (this is true remote from the ends).

(ii) Deformations are small.

The material is linear elastic, homogenous and isotropic.

10.3.1 Equilibrium



r, θ, z coordinate system

σ_r = radial stress

σ_θ = hoop stress

σ_z =axial stress

Figure 10.3: Internally pressurized thick cylinder and FBD of an element of material within the cylinder

$$\left(\sigma_r + \frac{d\,\sigma_r}{dr}\delta r\right)(r+\delta r)\delta\theta\delta z = \sigma_r(r\delta\theta\delta z) + 2\sigma_\theta(\delta r\delta z)\sin\left(\frac{\delta\theta}{2}\right)$$

For small *δθ*, $\sin\left(\frac{\delta\theta}{2}\right) \approx \frac{\delta\theta}{2}$ therefore:

$$\sigma_r(r+\delta r)\delta\theta + \frac{d\sigma_r}{dr}\delta r(r+\delta r)\delta\theta = \sigma_r r\delta\theta + \sigma_\theta\delta r\delta\theta$$

$$r\sigma_r + \sigma_r \delta r + r\frac{d\sigma_r}{dr}\delta r + \frac{d\sigma_r}{dr}\delta r^2 = \sigma_r r + \sigma_\theta \delta r$$

As $\delta r \to 0,\ \frac{d\sigma_r}{dr}\delta r^2 \to 0$

$$\sigma_\theta - \sigma_r = r\frac{d\sigma_r}{dr} \quad (32)$$

10.3.2 Compatibility

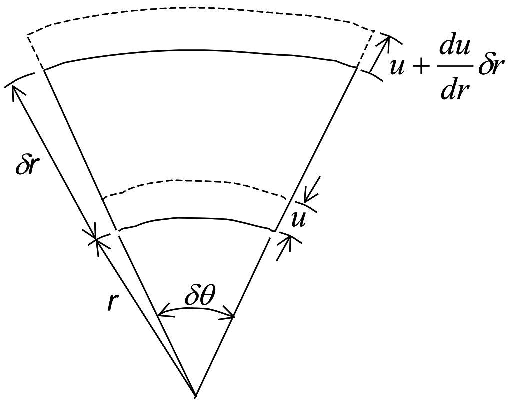


Figure 10.4: Initial and deformed shape of an element of material

$$\varepsilon = \frac{\text{extension}}{\text{original length}}$$

Hoop strain, $\varepsilon_\theta = \frac{(r+u)\delta\theta - r\delta\theta}{r\delta\theta} = \frac{u}{r}$ (33)

Radial strain, $\varepsilon_r = \frac{\left(u + \frac{du}{dr}\delta r\right) - u}{\delta r} = \frac{du}{dr}$ (34)

Axial strain, $\varepsilon_z = \text{constant}$ (35)

10.3.3 Material behaviour

Generalised Hooke's Law (linear elastic and isotropic)

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_z)) \tag{36}$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu(\sigma_\theta + \sigma_z)) \tag{37}$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_r + \sigma_\theta)) \tag{38}$$

Equations (3) to (9) have seven unknowns, i.e. u, σ_θ, σ_r, σ_z, ε_θ, ε_r and ε_z, which are all functions of r, p_o, p_i, R_o, R_i, ν and E.

Substituting $u = r\varepsilon_\theta$ from Eq. (4) into Eq. (5) gives

$$\varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta)$$

i.e. $\varepsilon_r = \varepsilon_\theta + r\dfrac{d\varepsilon_\theta}{dr}$ (a)

Using Eq. (7) and Eq. (8) in Eq. (a) gives

$$\frac{1}{E}(\sigma_r - \nu(\sigma_\theta + \sigma_z)) = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_z)) + \frac{r}{E}\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_z}{dr}\right)$$

$ie\,(1+\nu)\sigma_r = (1+\nu)\sigma_\theta + r\dfrac{d\sigma_\theta}{dr} - r\nu\dfrac{d\sigma_r}{dr} - r\nu\dfrac{d\sigma_z}{dr}$ (b)

Using Eq. (6), $\dfrac{d\varepsilon_z}{dr} = 0$, then Eq. (9) gives:

$$0 = \frac{d\sigma_z}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_\theta}{dr}$$

$\therefore\ \dfrac{d\sigma_z}{dr} = \nu\dfrac{d\sigma_r}{dr} + \nu\dfrac{d\sigma_\theta}{dr}$ (c)

Substituting (c) in (b)

$$(1+\nu)\sigma_r = (1+\nu)\sigma_\theta + r\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr} - r\nu^2\frac{d\sigma_r}{dr} - r\nu^2\frac{d\sigma_\theta}{dr}$$

$$ie(1+\nu)\sigma_r = (1+\nu)\sigma_\theta + r(1-\nu^2)\frac{d\sigma_\theta}{dr} - r\nu(1+\nu)\frac{d\sigma_r}{dr}$$

$$\therefore \sigma_\theta - \sigma_r = r\nu\frac{d\sigma_r}{dr} - r(1-\nu)\frac{d\sigma_\theta}{dr} \qquad \text{(d)}$$

Substituting the right hand side from Eq. (d) into Eq. (3) gives:

$$r\nu\frac{d\sigma_r}{dr} - r(1-\nu)\frac{d\sigma_\theta}{dr} = r\frac{d\sigma_r}{dr}$$

i.e. $$r(1-\nu)\left[\frac{d\sigma_r}{dr} + \frac{d\sigma_\theta}{dr}\right] = 0$$

$$\therefore \frac{d}{dr}(\sigma_r + \sigma_\theta) = 0$$

integration leads to: $\sigma_r + \sigma_\theta = 2A$ (constant of integration) (e)

but Eq (3) states: $\sigma_\theta - \sigma_r = r\frac{d\sigma_r}{dr}$

so substituting in to (e) for σ_θ $\quad 2\sigma_r = 2A - r\frac{d\sigma_r}{dr}$

and rearranging: $r\frac{d\sigma_r}{dr} + 2\sigma_r = 2A$

which is equivalent to: $\frac{1}{r}\frac{d}{dr}(r^2\sigma_r) = 2A$

Hence: $r^2\sigma_r = \frac{2Ar^2}{2} - B$

where *B* is another constant of integration

i.e. $$\boxed{\sigma_r = A - \frac{B}{r^2}}$$

and using Eq. (e) leads to $$\boxed{\sigma_\theta = A + \frac{B}{r^2}}$$

Note that, since $\varepsilon_z = \text{const}$ and $\sigma_r + \sigma_\theta = \text{const}$, then Eq. (9) shows that $\sigma_z = \text{const}$, i.e. it is independent of *r*. The value of σ_z can therefore be obtained from a consideration of axial equilibrium.

$$\sigma_r = A - \frac{B}{r^2}$$

and $$\sigma_\theta = A + \frac{B}{r^2}$$

The constants, *A* and *B*, are the so-called Lame's constants, which are the constants of integration, can be obtained from the boundary conditions, i.e.

at $r = R_i$, $\sigma_r = -p_i$

at $r = R_o$, $\sigma_r = -p_o$

$$\therefore \; -p_i = A - \frac{B}{R_i^2}$$

$$and \; -p_o = A - \frac{B}{R_o^2}$$

Hence, *A* and *B* can be determined

For closed-ended cylinders:

$$\pi(R_o^2 - R_i^2)\sigma_z + \pi R_o^2 p_o = \pi R_i^2 p_i$$

$$ie \; \sigma_z = \frac{R_i^2 p_i - R_o^2 p_o}{(R_o^2 - R_i^2)}$$

For a solid cylinder, i.e. $R_i = 0$

$\sigma_{r(r=R_i=0)} = A - \dfrac{B}{0^2} = \infty$, unless $B = 0$

Therefore, *B* must be zero, since the stresses cannot be infinite, and so, for a solid cylinder, the radial and hoop stresses are equal to each other and they are constant, i.e. $\sigma_r = \sigma_\theta = A$

Also, since a solid cylinder can only have external pressure, the constant *A* must equal the external pressure.

Displacements are most conveniently obtained by using Eqs. (7) and (9) together with Eqs. (4) and (6), i.e.

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_z))$$

$$\varepsilon_z = \frac{\Delta l}{l} = \frac{1}{E}(\sigma_z - \nu(\sigma_r + \sigma_\theta)) = \text{constant}$$

where l is the cylinder length, Δl is the increase in cylinder length and u is the radial displacement at radius r.

10.4 Analysis of rotating discs

Rotating components such as flywheels and turbine discs can be regarded as **thick cylinders with body forces**, as well as possible pressure loads and as such represent an extension of the thick cylinder theory discussed in the previous section.

10.4.1 Equilibrium

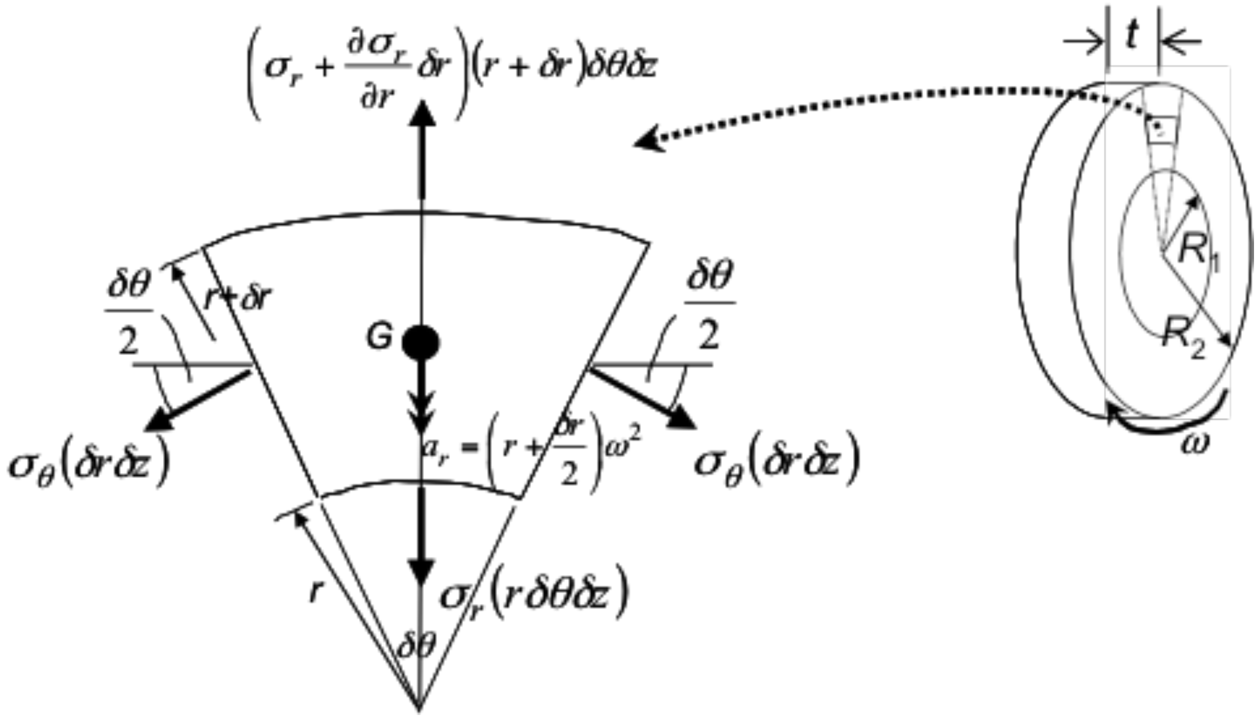


Figure 10.5: body diagram of an element of material within a disc

$$\sigma_r r\delta\theta\delta z + 2\sigma_\theta\delta r\delta z\left(\frac{\delta\theta}{2}\right) - \left(\sigma_r + \frac{d\sigma_r}{dr}\delta r\right)(r+\delta r)\delta\theta\delta z = \left[\rho\left(r+\frac{\delta r}{2}\right)\delta\theta\delta z\delta r\right]\left(r+\frac{\delta r}{2}\right)\omega^2$$

$$\sigma_r r + \sigma_\theta\delta r - r\sigma_r - \sigma_r\delta r - r\frac{d\sigma_r}{dr}\delta r - \frac{d\sigma_r}{dr}(\delta r)^2 = \rho\left(r+\frac{\delta r}{2}\right)^2\delta r\omega^2$$

$$\therefore \sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} - \frac{d\sigma_r}{dr}\delta r = \rho r^2\omega^2 + \rho\left(\frac{\delta r}{2}\right)^2\omega^2 + \rho r\delta r\omega^2$$

Neglecting small terms, i.e. those containing δr and $(\delta r)^2$

$$\boxed{\sigma_\theta - \sigma_r = r\frac{d\sigma_r}{dr} + \rho r^2\omega^2} \qquad (1)$$

also

$$\sigma_z = 0$$

since it is a disc with no applied axial forces and is not constrained axially along its faces.

10.4.2 Compatibility and material behaviour

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \qquad (2)$$

$$\varepsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \qquad (3)$$

Substituting for *u* from Eq. (2) into to Eq. (3) gives:

$$\frac{d}{dr}\left(\frac{r}{E}(\sigma_\theta - \nu\sigma_r)\right) = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$ie,\ \sigma_\theta - \nu\sigma_r + r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) = \sigma_r - \nu\sigma_\theta$$

$$\boxed{\therefore (\sigma_\theta - \sigma_r)(1+\nu) + r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) = 0} \qquad \text{(a)}$$

Substituting for $\sigma_\theta - \sigma_r$ from Eq. (1) into Eq. (a) gives:

$$\left(r\frac{d\sigma_r}{dr} + \rho r^2\omega^2\right)(1+\nu) + r\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr} = 0$$

$$\therefore r\frac{d\sigma_r}{dr} + r\nu\frac{d\sigma_r}{dr} + (1+\nu)\rho r^2\omega^2 + r\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr} = 0$$

$$ie, \frac{d}{dr}(\sigma_\theta + \sigma_r) = -(1+\nu)\rho\omega^2 r$$

$$\therefore \quad \sigma_\theta + \sigma_r = -(1+\nu)\rho\omega^2\frac{r^2}{2} + 2A \qquad \text{(b)}$$

where *B* is a constant of integration. Therefore,

$$\boxed{\sigma_r = A - \frac{B}{r^2} - \frac{\rho\omega^2(3+\nu)}{8}r^2}$$

and from Eq. (b):

$$\sigma_\theta = A + \frac{B}{r^2} + \frac{\rho\omega^2(3+\nu)}{8}r^2 - \frac{\rho\omega^2(1+\nu)}{2}r^2$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{\rho\omega^2(1+3\nu)}{8}r^2$$

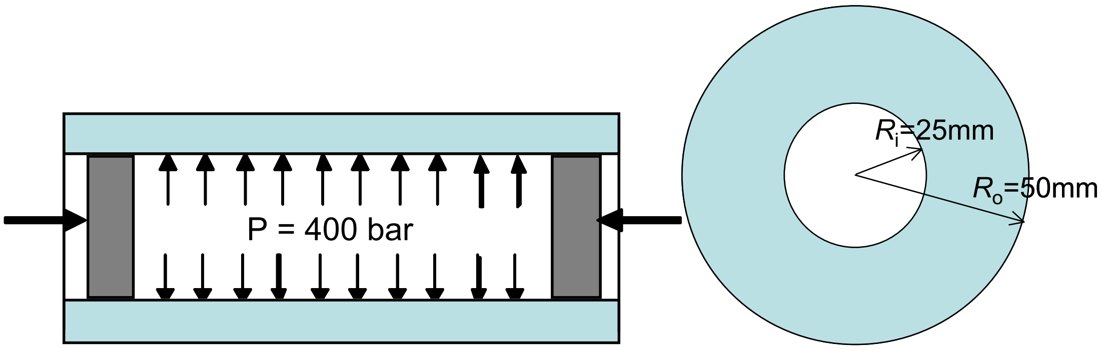
10.5 Example 1: Thick cylinder with pistons

A cylinder with 50m bore and 100mm OD is subjected to an internal pressure of 400bar. The end loads are supported by pistons, which seal without restraint. Determine the distributions of stress across the cylinder wall.

P = 400 bar = 400 X 100 kPa

= 40 X 1000 kPa

= 40 N/mm^2



Since there is no axial load on the cylinder, then => σ_z = 0.

For a thick cylinder:

$$\sigma_r = A - \frac{B}{r^2} \tag{1}$$

$$\sigma_\theta = A + \frac{B}{r^2} \tag{2}$$

Using the boundary conditions:

At r = 25mm, σ_r = -40 N/mm^2, therefore:

$$40 = A - \frac{B}{635} \tag{3}$$

At r = 50mm, σ_r = 0

$$0 = A - \frac{B}{2500} \tag{4}$$

Rearrange to give:

$$A = \frac{B}{2500} \qquad (5)$$

Substituting equation (5) in to equation (3) eliminates A and gives:

$$40 = B\left(\frac{1}{625} - \frac{1}{2500}\right)$$

or:

$$40 = B\left(\frac{4-1}{2500}\right)$$

therefore:

$$B = \frac{4 \times 2500}{3} \qquad (6)$$

Substituting for B into equation (4) gives:

$$-40 = A - \frac{40 \times 2500}{3 \times 625}$$

therefore:

$$A = \frac{40}{3} \qquad (7)$$

$$-40 = A - \frac{40 \times 2500}{3 \times 625}$$

$$\therefore A = \frac{40}{3}$$

Hence,

$$\sigma_\theta = \frac{40}{3} + \frac{40 \times 2500}{3r^2} = \frac{40}{3}\left(1 + \frac{2500}{r^2}\right)$$

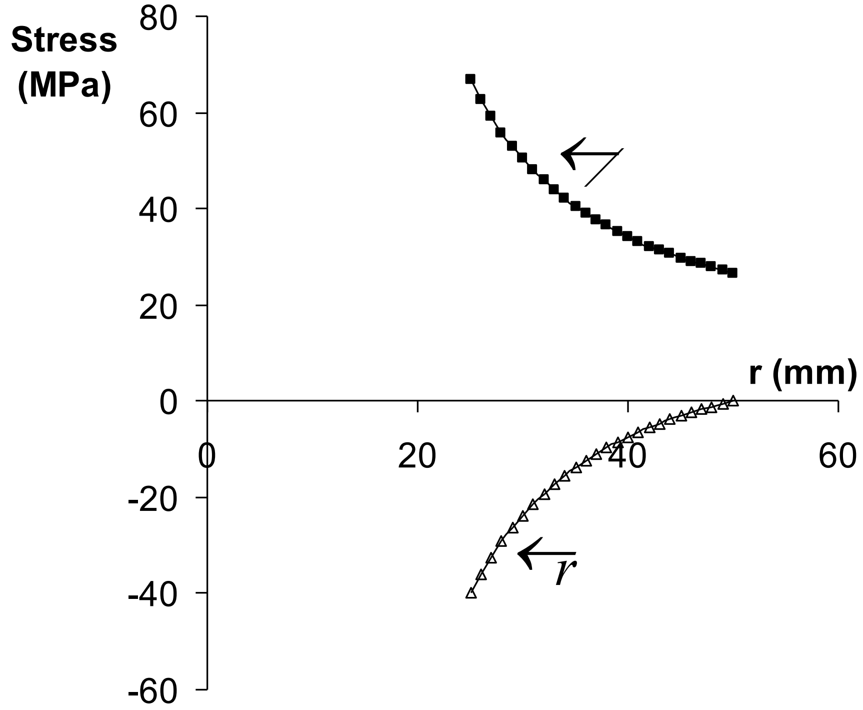
and $\sigma_r = \frac{40}{3} - \frac{40 \times 2500}{3r^2} = \frac{40}{3}\left(1 - \frac{2500}{r^2}\right)$

At r = 25mm, $\sigma_\theta = \frac{40}{3} \times 5\text{N/mm}^2 = 66.7\text{N/mm}^2$

and $\sigma_r = \frac{40}{3} \times (-3)\text{N/mm}^2 = -40\text{N/mm}^2$

At r = 50mm, $\sigma_\theta = \frac{40}{3} \times 2\text{N/mm}^2 = 26.7\text{N/mm}^2$

and $\sigma_r = 0$

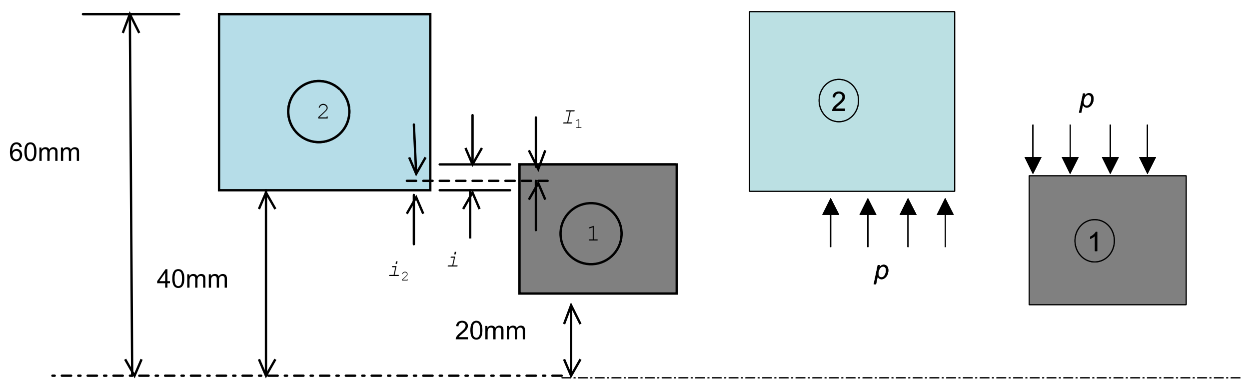


10.6 Example 2: Shrink/interference fit

A pair of mild steel cylinders (E = 200 GPa) of equal length have the following dimensions:

1. 40mm bore and 80.06mm outside diameter
2. 80mm bore and 120mm outside diameter

i.e. there is a diametral interference of 0.06mm. The larger cylinder is heated, placed around and allowed to shrink onto the smaller cylinder. Calculate the stresses after assembly.



Conditions

(i) after assembly, the radial interference pressure, p, will be the same on both cylinders, i.e. Cylinder 1 will have an external pressure, p, and Cylinder 2 will have an internal pressure, p, as indicated in the above figure.

(ii) The decrease in the outside radius of Cylinder 1, i_1, plus the increase in the inside radius of Cylinder 2, i_2, will be equal to the radial interference, i.e. $i = i_1 + i_2$

(iii) axial stresses are assumed to be zero (or negligible)

For cylinder (1):

$$\sigma_r = A_1 - \frac{B_1}{r^2}$$

and $\sigma_\theta = A_1 + \dfrac{B_1}{r^2}$

at r = 20mm, σ_r = 0,

$\therefore B_1 = 400A_1$

at r = 40mm (no significant difference with 40.03mm), σ_r = -p

$$\therefore -p = A_1 - \frac{20^2}{40^2}A_1 = A_1 - \frac{400}{1600}A_1$$

$$ie\ A_1 = -\frac{4}{3}p$$

$$and\ B_1 = -\frac{1600}{3}p$$

Thus, $(\sigma_r)_1 = -\frac{4p}{3}\left(1 - \frac{400}{r^2}\right)$

and $(\sigma_\theta)_1 = -\frac{4p}{3}\left(1 + \frac{400}{r^2}\right)$

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_z)) = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

At the outside of cylinder (1), r = 40mm,

$$\frac{-i_1}{40} = \frac{1}{200{,}000}(\sigma_\theta - \nu\sigma_r)$$

$$ie\ \frac{-i_1}{40} = \frac{1}{200{,}000}\left(-\frac{4p}{3}\right)\left(1 + \frac{400}{1600} - \nu\left(1 - \frac{400}{1600}\right)\right)$$

$$i_1 = \frac{8p}{30000}\left(\frac{5}{4} - \frac{3\nu}{4}\right)$$

$$\therefore i_1 = \frac{2p}{30000}(5 - 3\nu)$$

For cylinder (2):

$$\sigma_r = A_2 - \frac{B_2}{r^2}$$

and $\sigma_\theta = A_2 + \frac{B_2}{r^2}$

At r = 60mm, σ_r = 0

$\therefore B_2 = 3600A_2$

At r = 40mm, σ_r = -p

$$\therefore -p = A_2 - \frac{60^2}{40^2} A_2 = A_2 - \frac{3600}{1600} A_2$$

$$ie\ A_2 = \frac{4}{5} p$$

$$and\ B_2 = 3600 \times \frac{4}{5} p$$

Thus, $(\sigma_r)_2 = \frac{4p}{5}\left(1 - \frac{3600}{r^2}\right)$

and $(\sigma_\theta)_2 = \frac{4p}{5}\left(1 + \frac{3600}{r^2}\right)$

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right) = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

At the inside of cylinder (2), r = 40mm,

$$\frac{+i_2}{40} = \frac{1}{200{,}000}\left(\frac{4p}{5}\right)\left(1 + \frac{3600}{1600} - \nu\left(1 - \frac{3600}{1600}\right)\right)$$

$$ie\ i_2 = \frac{8p}{50000}\left(\frac{13}{4} + \frac{5\nu}{4}\right)$$

$$\therefore i_2 = \frac{2p}{50000}\left(13 + 5\nu\right)$$

But $i_1 + i_2 = i$ = 0.03mm

$$\therefore \frac{2p}{30000}\left(5 - 3\nu\right) + \frac{2p}{50000}\left(13 + 5\nu\right) = 0.03$$

$$\frac{10p}{30000} - \frac{2\nu p}{10000} + \frac{26p}{50000} + \frac{2\nu p}{10000} = 0.03$$

$$\frac{50p + 78p}{150{,}000} = 0.03$$

$$ie\ p = \frac{4500}{128}\ \text{N/mm}^2 = 35.2\ \text{N/mm}^2$$

For cylinder (1),

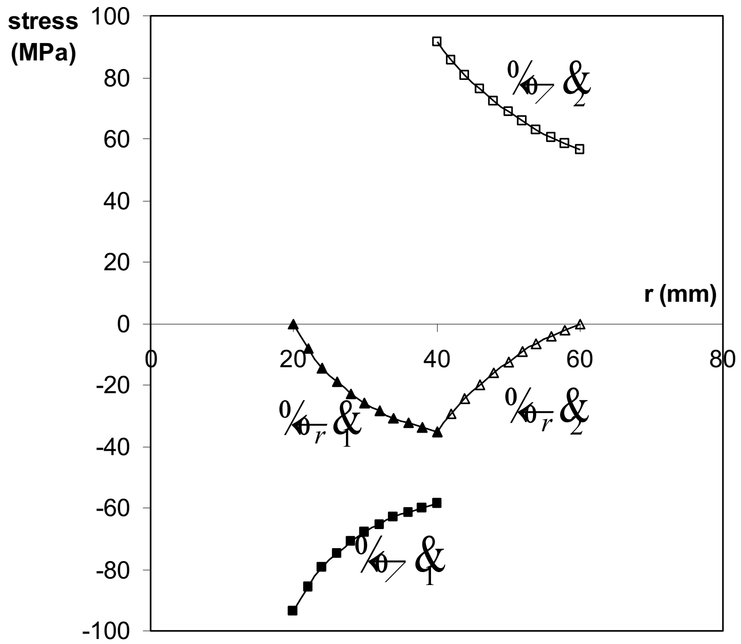
$$(\sigma_r)_1 = -46.9\left(1 - \frac{400}{r^2}\right)$$

$$(\sigma_\theta)_1 = -46.9\left(1+\frac{400}{r^2}\right)$$

and for cylinder (2),

$$(\sigma_r)_2 = 28.2\left(1-\frac{3600}{r^2}\right)$$

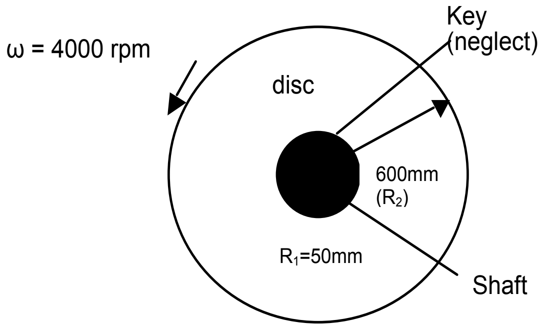
$$(\sigma_\theta)_2 = 28.2\left(1+\frac{3600}{r^2}\right)$$



10.7 Example 3: Turbine disc

A turbine rotor disc with an angular velocity of 4000rpm has an external diameter of 1.2m and has a 0.1m diameter hole bored along its axis. Determine the stress distributions in the disc.

Take: $\rho = 7850 kg/m^3 and \;\; \nu = 0.3$



$$\sigma_r = A - \frac{B}{r^2} - \frac{\rho\omega^2(3+\nu)r^2}{8} \tag{8}$$

$$\sigma_\theta = A - \frac{B}{r^2} + \frac{\rho\omega^2(1+3\nu)r^2}{8} \tag{9}$$

at 50mm, σ_r= 0, therefore:

$$0 = A - \frac{B}{50^2} - \frac{7850 \times 10^{-9} \times (4000 \times \left(\frac{2\pi}{60}\right) \times (3+0.3) \times 50^2 \times 10^{-3}}{8}$$

(note that the 10^{-3} is required for consistency of units when working in mm as the body force is required in N)

gives:

$$0 = A - \frac{B}{2500} - 1.4204 \tag{10}$$

at r = 600mm, σ_r= 0, therefore:

$$0 = A - \frac{B}{600^2} - \frac{7850 \times 10^{-9} \times (4000 \times \left(\frac{2\pi}{60}\right) \times (3 + 0.3) \times 600^2 \times 10^{-3}}{8}$$

gives:

$$0 = A - \frac{B}{3.6 \times 10^5} - 204.54 \qquad (11)$$

subtracting (10) from (11):

$$B\left(\frac{1}{2500} - \frac{1}{3.6 \times 10^6}\right) = 204.54 - 1.42$$

therefore:

B = 511350

and

A = 205.95

Hence,

$$\sigma_r = 205.95 - \frac{511350}{r^2} - 5.68 \times 10^{-4} r^2 \qquad (12)$$

and

$$\sigma_\theta = 205.95 + \frac{511350}{r^2} - 3.27 \times 10^{-4} r^2 \qquad (13)$$

Evaluating (13) at the inner and outer diameters:

At r = 50mm, σ_θ = 410 MPa

At r = 600mm, σ_θ = 89.6 MPa

The variation of both quantities can be seen in Table 1 and Figure 10.6

Table 1: Hoop and radial stress values

	r [mm]	sigma_r [MPa]	sigma_theta [MPa]
ri	50	0.0	409.7
	77.5	117.4	289.1
	105	153.3	248.7
	132.5	166.9	229.3
	160	171.4	217.6
	187.5	171.4	209.0
	215	168.6	201.9
	242.5	163.9	195.4
	270	157.5	189.1
	297.5	149.9	182.8
	325	141.1	176.2
	352.5	131.2	169.4
	380	120.4	162.3
	407.5	108.5	154.7
	435	95.7	146.8
	462.5	82.0	138.4
	490	67.4	129.5
	517.5	51.9	120.3
	545	35.5	110.5
	572.5	18.2	100.3
ro	600	0.0	89.6

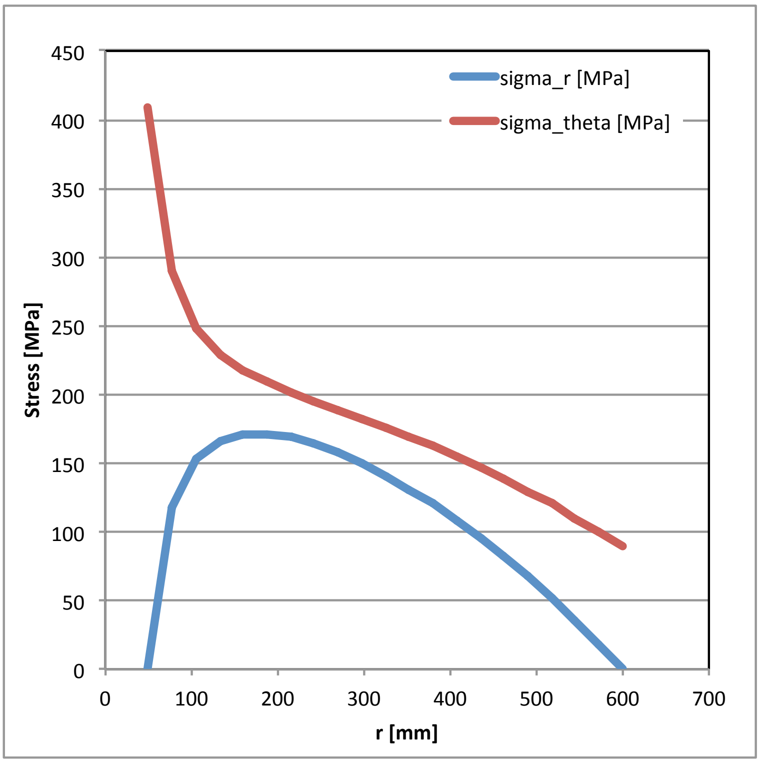


Figure 10.6: Hoop and radial stress variation

The maximum value of σ_r can be obtained by using

$$\frac{d\sigma_r}{dr} = 0$$

to find the value of r and then evaluating (12) at this location. Leading to a value of 172MPa at a diameter of 173mm.