7 Shear Stresses in Beams

Learning Summary

1. Appreciate that in addition to longitudinal bending stresses, beams also carry transverse shear stresses arising from the vertical shear loads acting within the beam (knowledge)
2. Be able to derive a general formula, in both integral and discrete form, for evaluating the shear stress distribution through a cross-section (comprehension);
3. Determine the shear stress distribution through the thickness in a rectangular, circular and I-section beam (application);
4. Understand that in an I-section, in addition to the transverse vertical shear stresses in the flange and web, more dominant horizontal shear stresses also occur in the flange (comprehension);
5. Recognise that the resultant of the shear stresses always act through one point, known as the 'shear centre' (comprehension);
6. Calculate the position of the shear centre (application);
7. Understand that if the applied loads do not act through the shear centre, then there is a resultant torsional load, which can result in twisting of the section if the torsional rigidity of the section is low e.g. thin walled sections (comprehension).

7.1 Introduction

Whereas bending stresses in beams arising from transverse loading are important, transverse (i.e. through-thickness) shear stresses due to these same loads also exist. For long slender beams, the shear stresses can generally be neglected, and it is only necessary to do a bending calculation for the beam. However, as the beam span to depth ratio reduces, i.e. if the beam is shorter and thicker, shear stresses become more important and should be calculated in any design evaluation. This can be particularly important for laminated beams, e.g. plywood or composite beams, where the transverse shear can cause failure between individual layers (plies) making up the beam. In this section we will derive a general formula for calculating the shear stress distribution through the thickness of a beam. We will then introduce the concept of shear centre, which is the point through which the resultant of the shear stresses always acts. The shear centre becomes important for beam sections with low torsional rigidity, i.e. can twist easily, such as thin-walled sections. For such beams, if the resultant of the applied transverse loads does not act through the shear centre, they can cause twisting of the beam i.e. there is a bending-twisting interaction in the system. The designer should avoid this situation if possible or, at least, evaluate the degree of twisting which might take place.

7.2 Transverse shear stress distribution

The through-thickness shear force in a beam is the integral of the shear stresses over the cross-section. In this section we will determine an expression for the shear stress distribution (transverse i.e. through-thickness) at a section as a function of the shear force at that position. Consider an element of beam length, δx, as shown in Figure 7.1. The bending moment at x, section AC, is M and at x + δx, section BD, is M + δM. The direct bending stresses on AC are,

$$\sigma_{AC} = \frac{My}{I}$$

Wherey = distance from the neutral surface/axis

I = 2^{nd} moment of area of the section

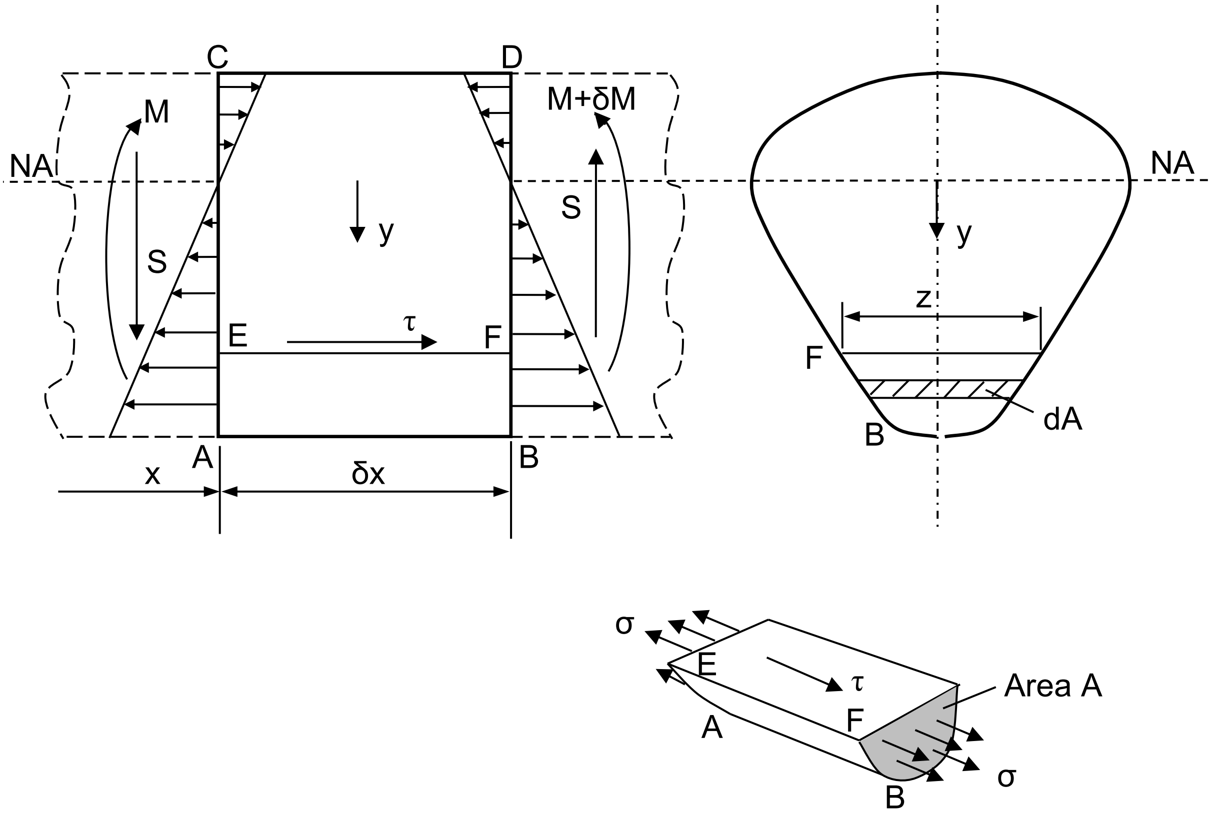


Figure 7.1

and on BD, the bending stresses are,

$$\sigma_{BD} = \frac{(M + \delta M)y}{I}$$

Thus, when the bending moment varies along the length of the beam on an element such as ABEF, also shown in Figure 7.1, there is a net axial force due to change in the bending stresses. The force on the face FB is the integral of the bending stresses over the area FB,

$$F_{FB} = \int_A \frac{(M + \delta M)}{I} y\, dA$$

Similarly the force on the face EA is,

$$F_{EA} = \int_A \frac{M}{I} y \, dA$$

The net force to the right acting on the element ABEF is the difference in these, i.e.,

$$\text{Net Force (bending)} = \int_A \frac{\delta M}{I} y \, dA \quad [1]$$

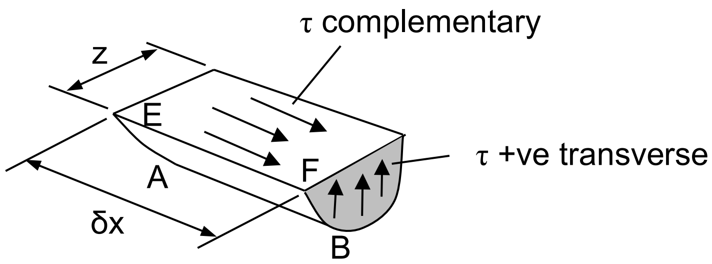


Figure 7.2

In order to maintain equilibrium of ABEF, shear stresses must act on the plane EF, of average value τ, as shown in Figure 7.2. These shear stresses are complementary to the <u>transverse</u> shear stresses. For positive transverse shear stresses, as shown, the complementary shear stresses act in the positive x direction. The net force to the right due to these complementary shear stresses is,

$$\text{Net Force (shear)} = \tau . z . \delta x$$

where z is the width of the section at that depth

Now, equilibrium of ABEF requires the net force due to bending to balance the net force due to the complementary shear. Thus,

$$\tau \, z \, \delta x + \int_A \frac{\delta M}{I} y \, dA = 0$$

$$\therefore \tau = -\frac{1}{I\,z}\frac{\delta M}{\delta x}\int_A y \, dA$$

But, in the limit, $\dfrac{\lim}{\delta x \to 0} \quad \dfrac{\delta M}{\delta x} = \dfrac{dM}{dx} = -S$

where S = shear force at the section

$$\therefore \tau = \frac{S}{I\,z}\int_A y \, dA \qquad [2]$$

This is the general expression for transverse shear stress at any position y through the thickness. The integral can also be written in discrete form as follows,

$$\tau = \frac{S}{I\,z} A \, \bar{y} \qquad [3]$$

where A is the area of the part of the cross-section outside the position at which τ is determined, and $\bar{y}$ is the distance of the centroid of this area from the neutral axis, as shown in Figure 7.3.

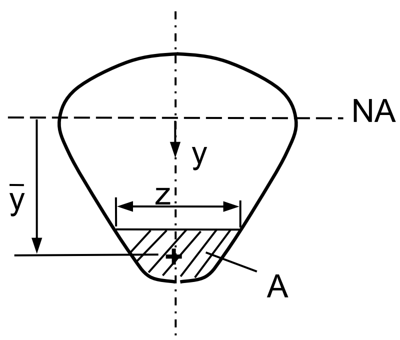


Figure 7.3

7.3 Determination of shear stress distribution for different cross-sectional shapes

7.3.1 Rectangular section

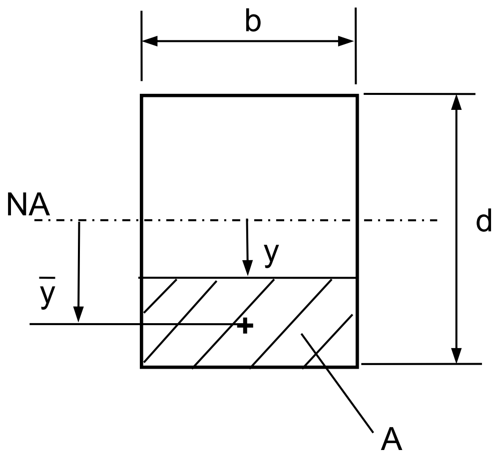


Figure 7.4

Referring to Figure 7.4 and using the discrete form for shear stress distribution, i.e. equation 3 above, we have,

$$A = \left(\frac{d}{2} - y\right)b \qquad \text{and} \qquad \bar{y} = \left(\frac{d}{2} + y\right)\frac{1}{2}$$

$$\tau = \frac{S}{(bd^3/12).b}\left(\frac{d}{2} - y\right)b\left(\frac{d}{2} + y\right)\frac{1}{2}$$

$$\therefore \tau = \frac{6S}{bd^3}\left[\left(\frac{d}{2}\right)^2 - y^2\right] \qquad [4]$$

Note the parabolic distribution of shear stress (i.e. τ varies with y^2), illustrated in Figure 7.5. Also, at the top and bottom of the section, where $y = \pm d/2$, equation 4 gives $\tau = 0$. As expected, there is no complementary shear stress on the top and bottom free surfaces, therefore the transverse shear stress is also zero at these positions.

At the neutral axis, i.e. where y = 0, equation 4 gives,

$$\tau = \frac{6S}{bd^3}\frac{d^2}{4} = 1.5\frac{S}{bd}$$

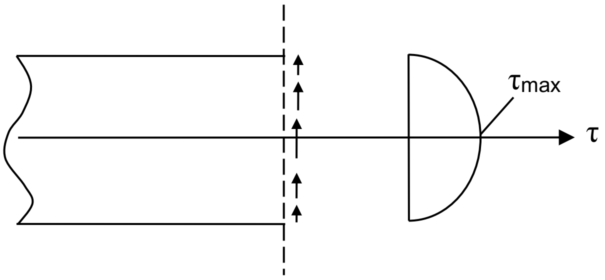


Figure 7.5

This is the position of maximum shear stress whose magnitude is 1.5X the average shear stress S/bd.

Note also that, in this analysis, the shear stress does not vary across the width of the section.

7.3.2 Circular section

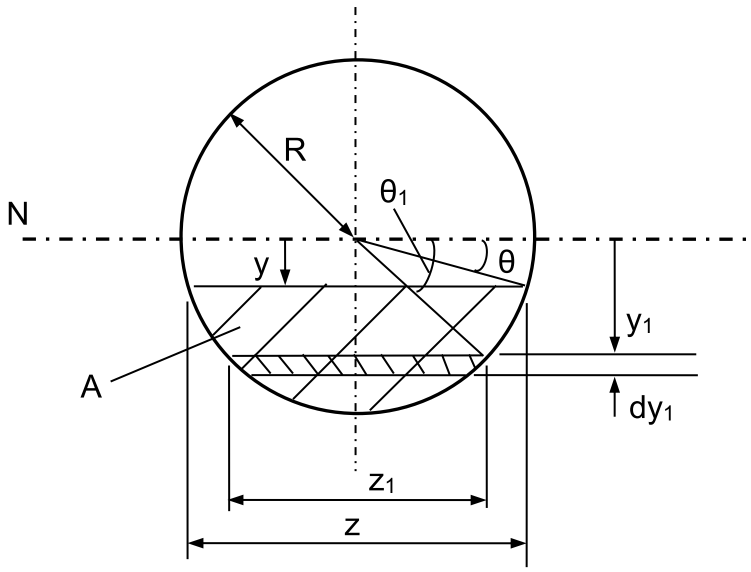


Figure 7.6

Figure 7.6 shows a solid circular cross-section of a beam. To calculate the transverse shear stress distribution in this section, we use the integral form of the shear equation i.e. equation 2. However, because of the circular shape, it is convenient to change the variables y and z in this equation to polar variables, R and θ. Referring to Figure 7.6 we have,

$$y_1 = R \sin\theta_1$$
$$dy_1 = R \cos\theta_1 d\theta_1$$
$$z_1 = 2R \cos\theta_1$$
$$z = 2R \cos\theta$$

and the 2nd moment of area, $I = \pi D^4/64 = \pi R^4/4$

The shear equation now becomes,

$$\tau = \frac{S}{I\,z}\int y\,dA = \frac{S.4}{\pi R^4.2R\cos\theta}\int_{\theta}^{\pi/2} y_1 z_1 dy_1$$

$$= \frac{S.4}{\pi R^4.2R\cos\theta}\int_{\theta}^{\pi/2} R_1 \sin\theta_1.2R\cos\theta_1.R\cos\theta_1 d\theta_1$$

$$= \frac{4SR^3}{\pi R^5 \cos\theta}\int_{\theta}^{\pi/2} \cos^2\theta_1.\sin\theta_1 d\theta_1$$

$$= \frac{4S}{\pi R^2 \cos\theta}\left[\frac{-\cos^3\theta_1}{3}\right]_{\theta}^{\pi/2}$$

$$\therefore \tau = \frac{4S}{3\pi R^2}\cos^2\theta$$

But $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{y}{R}\right)^2$

$$\therefore \tau = \frac{4S}{3\pi R^2}\left[1 - \left(\frac{y}{R}\right)^2\right] \qquad [5]$$

Again, a parabolic distribution and the maximum value of τ, at the neutral axis, when y=0 is,

$$\therefore \tau_{\max}(at\ y=0) = \frac{4S}{3\pi R^2} = \frac{4}{3}\tau_{average}$$

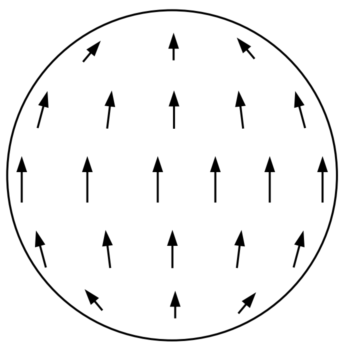


Figure 7.7

In this case, τ must vary across the width of the section. As can be seen in Figure 7.7, at the free surface the shear stress must be zero. Therefore, the complementary shear on the cross-section, normal to the boundary, is also zero. Thus, shear must be tangential to the boundary as drawn.

7.3.3 I-section

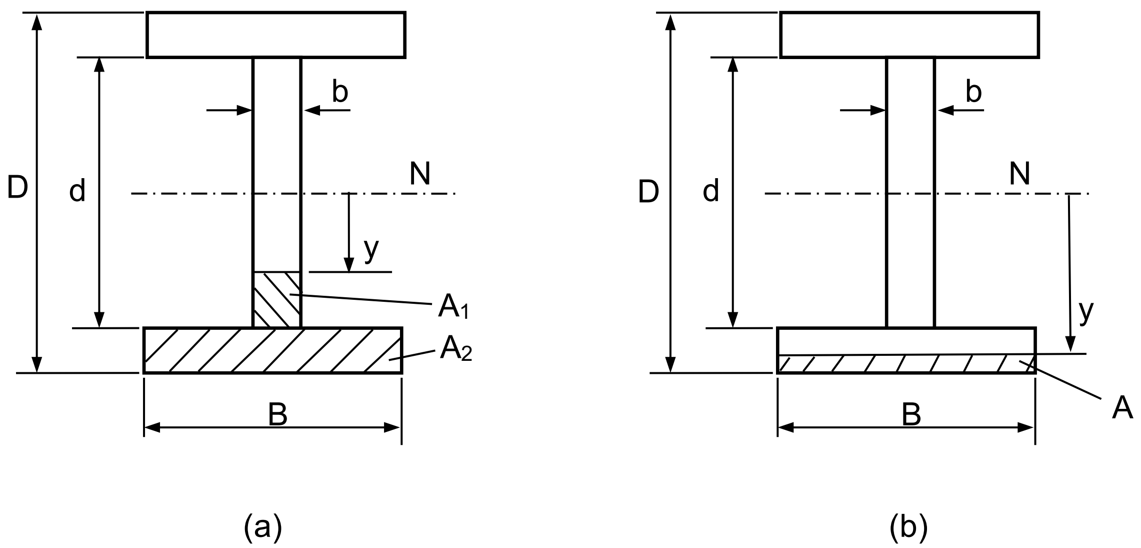


Figure 7.8

To determine the transverse shear stress distribution in an I-section, we need to consider the web and flange areas separately.

(i) Transverse shear in the web

Figure 7.8(a) shows an I-section and the position y where we wish to determine the shear stress. Using the discrete form of the shear stress equation we have,

$$\tau = \frac{S}{I\,z} A\bar{y} = \frac{S}{I\,z}[A_1\bar{y}_1 + A_2\bar{y}_2]$$
$$= \frac{S}{I\,b}\left[\left(\frac{d}{2} - y\right)\cdot b \cdot \frac{1}{2}\cdot\left(\frac{d}{2} + y\right) + B\left(\frac{D}{2} - \frac{d}{2}\right)\cdot\frac{1}{2}\cdot\left(\frac{D}{2} + \frac{d}{2}\right)\right]$$
$$= \underline{\frac{S}{I\,b}\left[\frac{b}{2}\cdot\left(\frac{d^2}{4} - y^2\right) + \frac{B}{2}\cdot\left(\frac{D^2}{4} - \frac{d^2}{4}\right)\right]}$$

and $\quad I = \dfrac{BD^3}{12} - \dfrac{(B-b)d^3}{12}$

The maximum τ at y=0: $\quad \tau_{\max} = \dfrac{S}{I\,b}\left[\dfrac{BD^2}{8} - \dfrac{(B-b)d^2}{8}\right]$

At the bottom and top of the web, where y=±d/2: $\quad \tau = \dfrac{S}{I\,b}\cdot\dfrac{B}{8}\cdot\left(D^2 - d^2\right) \qquad$ [6]

(ii) Transverse shear in the flange

Figure 7.8(b) shows the position y where we wish to determine the shear stress in the flange. Again, using the discrete form of the shear stress equation we have,

$$\tau = \frac{S}{I\,z} A\bar{y}$$
$$= \frac{S}{I\,b}\left[B\cdot\left(\frac{D}{2} - y\right)\cdot\frac{1}{2}\cdot\left(\frac{D}{2} + y\right)\right]$$
$$= \underline{\frac{S}{2I}\left(\frac{D^2}{4} - y^2\right)}$$

At y = D/2 $\quad$ τ = 0, as expected i.e. zero shear complementary to the free surface

At y = d/2 $\quad \tau = \dfrac{S}{8I}\left(D^2 - d^2\right)$. Comparing this expression with equation 6, there is a step change in τ from the web to the flange of magnitude B / b i.e. the ratio of the flange width to the web width.

Figure 7.9 shows the transverse shear stress distribution down the centre line of the section and illustrates the step change discussed above. The shear in the flanges is small compared to the web and the shear stress in the web is approximately uniform with vertical position. Because of the small shear in the flanges, the average shear stress in the web is ≈ S / bd i.e. the shear force divided by the area of the web.

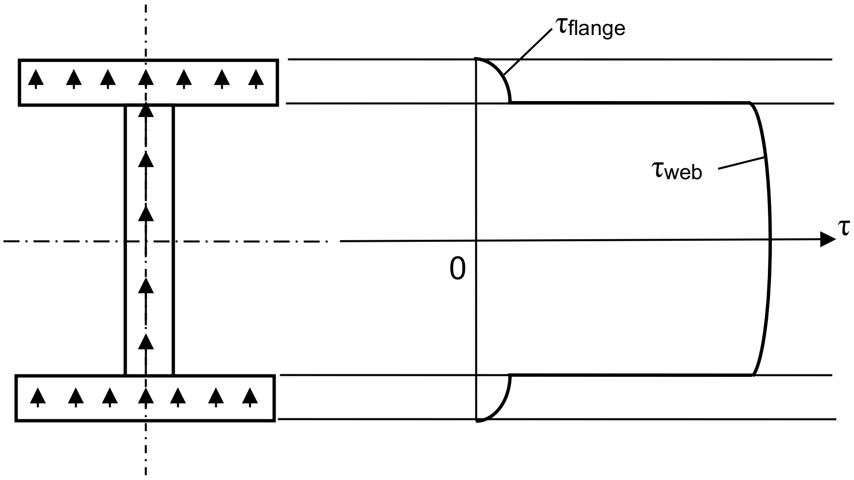


Figure 7.9

The above distribution only applies down the centre line of the web. The shear stresses in the flanges are small and non-uniform across the width. This must be the case as they must be zero at the top and bottom surfaces (i.e. free surfaces) of the flanges. There are, however, more significant shear stresses in the flanges which act parallel to the flanges i.e. horizontally. These can be determined by a similar analysis as follows:

(iii) Horizontal shear in the flange

Figure 7.10 shows a small length, δx, of I-beam over which the bending moment changes from M to M+δM. To determine the hidden horizontal shear stress, τ, at distance a from the edge of the flange, equilibrium of an element of the flange is considered. Equilibrium of stresses acting on the element gives,

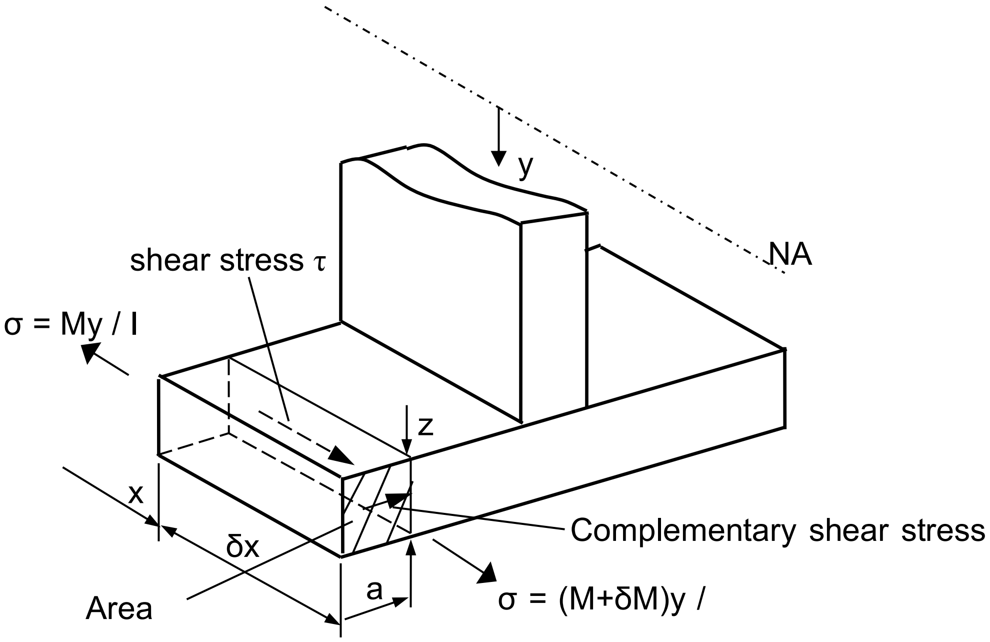


Figure 7.10

$$\int_A \frac{(M+\delta M)}{I} y\,dA - \int_A \frac{M}{I} y\,dA + \tau\, z\, \delta x = 0$$

$$\therefore \tau = -\frac{1}{I\,z}\frac{\delta M}{\delta x}\int_A y\,dA$$

In the limit as δx –> 0, $\frac{\delta M}{\delta x} = \frac{dM}{dx} = -S$

$$\underline{\therefore \tau = -\frac{S}{I\,z}\int_A y\,dA = \frac{S}{I\,z} A\,\bar{y}}$$

This is the same shear equation as before except the interpretation of the quantities A, $\bar{y}$ and z is different as shown in Figure 7.10. At a distance a from the edge of the flange, the horizontal shear stress is given by,

$$\therefore \tau = \frac{S}{I\,z}\cdot(a\,z)\cdot\frac{1}{2}\left(\frac{D}{2}+\frac{d}{2}\right)$$
$$= \underline{\frac{S\,a}{4I}(D+d)}$$

τ therefore varies linearly with a from zero at the flange edge to a maximum value at the flange centre (a=B/2),

$$\tau_{\max} = \frac{S\,B}{8I}(D+d)$$

τ is also parallel to the flange i.e. horizontal.

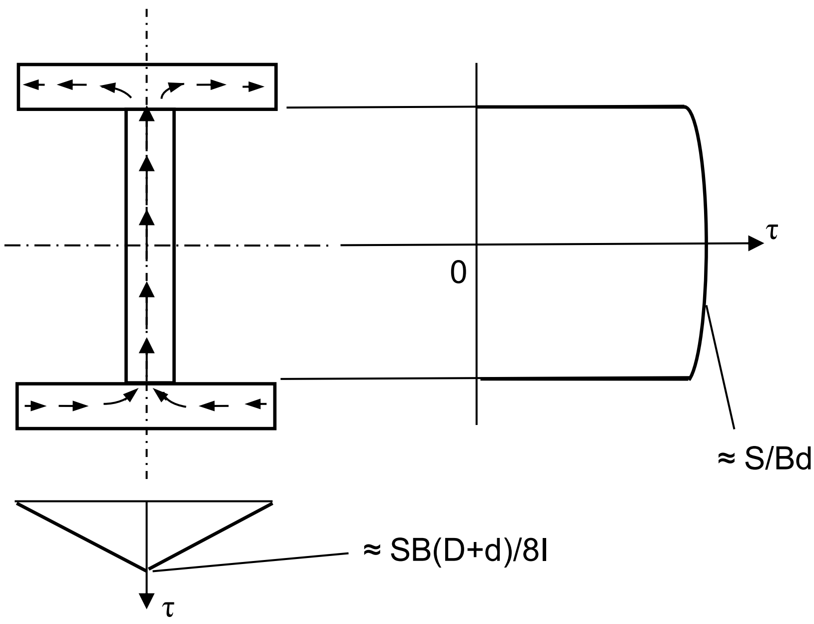


Figure 7.11

We can now draw the dominant shear stresses in both the flange and the web. Figure 7.11 shows the distribution of these horizontal and vertical (transverse) shear stresses. The critical stress position is likely to be at the join of the web and flange where both the shear and bending stresses are high.

7.4 Shear Centre

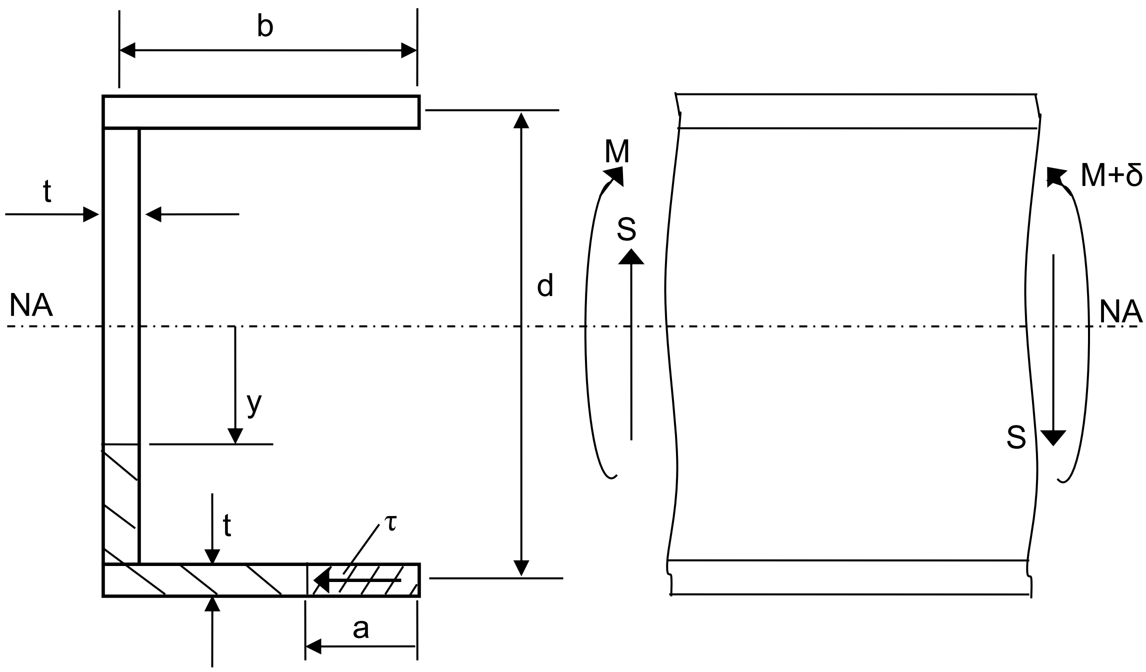


Figure 7.12

Consider the shear stress distribution in a symmetric, thin walled channel section bending in the plane of the web as shown in Figure 7.12.

For the flange at distance a from the edge, the horizontal shear stresses are,

$$\tau = \frac{S}{I\,z}A\bar{y} = \frac{S}{I\,t}(at)\cdot\left(\frac{d}{2}\right) = \frac{S.d.a}{2I}$$

[analysed as previously for the flange in an **I**-section]

For the web at distance y from the neutral axis, the transverse shear stresses are,

$$\tau = \frac{S}{I\,z} A\bar{y} = \frac{S}{I\,t}\left[bt\frac{d}{2} + \left(\frac{d}{2} - y\right)t\left(\frac{d}{2} + y\right)\frac{1}{2} \right]$$
$$= \frac{S}{2I}\left(bd + \left(\frac{d}{2}\right)^2 - y^2 \right)$$

We can now draw the shear stress distribution in both the flanges and the web, as shown in Figure 7.13(a). For this shear stress distribution, note that the shear stress in the upper flange is in the opposite sense to that in the lower flange i.e. there is no horizontal resultant. Also, as there are no shear stresses on the free surfaces, the shear stresses act along the walls i.e. horizontal in the flanges and vertical in the web.

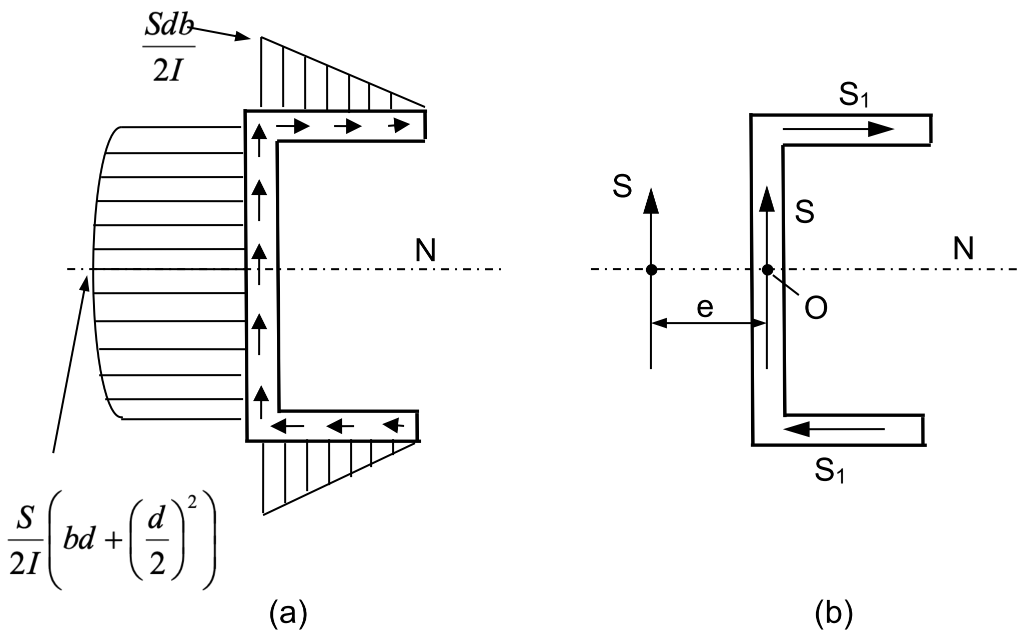


Figure 7.13

We can now look at the resultant forces arising from this shear stress distribution as shown in Figure 7.13(b).

The total shear force in the lower flange, S_1, is the integral of the shear stresses in this flange as follows,

$$S_1 = \int_0^b \tau\, t\, da = \int_0^b \frac{S\, d\, a}{2I}\, t\, da$$
$$= \underline{\frac{S d t b^2}{4I}}$$

An equal and opposite shear force acts in the upper flange.

The shear force in the web is approximately S i.e. the total vertical shear load [assuming thin flanges carry negligible vertical shear load].

The resultant of all the shear stresses must be the vertical shear force S, and its line of action is distance e outside the web. Now taking moments about O in the web,

$$S.e = 2S_1 \frac{d}{2}$$

$$\therefore e = \frac{S_1 d}{S} = \underline{\frac{d^2 t b^2}{4I}}$$

It can be shown that the resultant of the shear stresses for a section, for bending in any plane, always act through one point, the SHEAR CENTRE. The shear centre always lies on an axis of symmetry. For sections with two axes of symmetry, the shear centre is at the centroid.

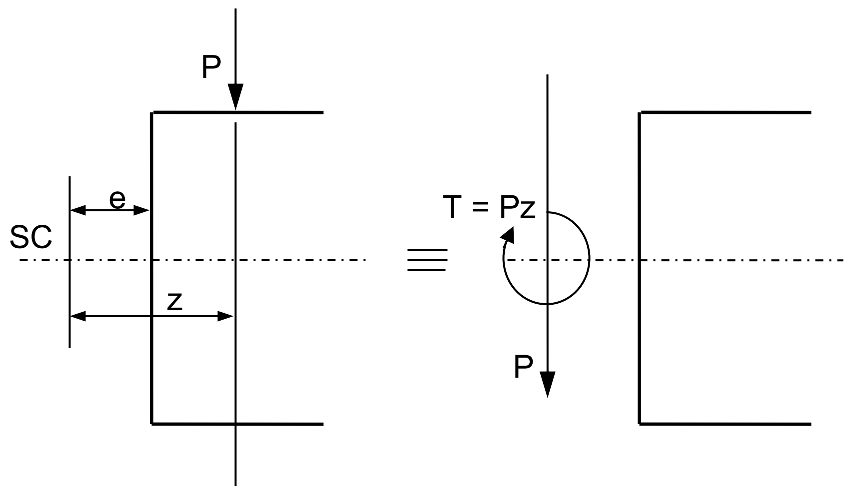
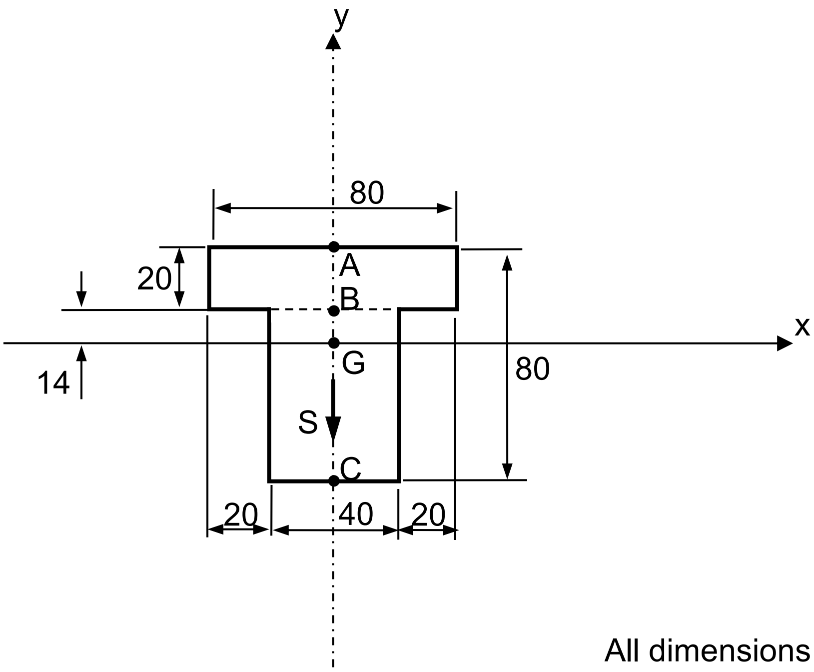


Figure 7.14

If the applied vertical loads do not act in the plane of the resultant of the shear stresses i.e. through the shear centre, then there is a torsional load on the section as shown in Figure 7.14. For arbitrary solid sections, the location of the shear centre is a complicated problem. However, it is <u>not</u> usually important to determine the shear centre for solid sections because such sections usually have a considerable torsional rigidity and twist very little due to bending loads. However, for thin-walled open sections, which have low torsional rigidity, the position of the shear centre may be very important.

7.5 Worked Examples

7.5.1 Shear Stresses in a Beam



All dimensions in mm

Figure 7.15

The section shown in Figure 7.15 is subjected to a vertical shear force, S = 50 kN, acting down the vertical centre line i.e. the y-axis. The second moment of area of the section, about the x-axis, which passes through the centroid of area, G, is $I_{xx} = 2.31 \times 10^6$ mm^4. G is positioned 14 mm below the flange.

(a) Determine the magnitude of the transverse (i.e. vertical) shear stress at positions A, B, G and C on the vertical centre line.

(b) Sketch the variation of the transverse shear stress down the vertical centre line

In the top flange:

Consider a position in the top flange, vertical distance y from the centroid, G, as shown in Figure 7.16(a). Using the discrete form of the shear formula,

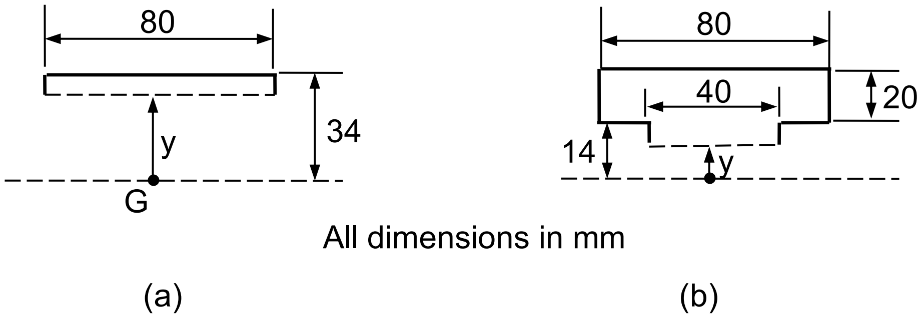


Figure 7.16

$$\tau = \frac{S}{I\,z} A\,\bar{y} = \frac{S}{2.31x10^6.80}.(80.(34-y)).\frac{(34+y)}{2}$$
$$= \frac{S}{4.62x10^6}(1156-y^2)$$
$$= 0.0108(1156-y^2)$$

At position A, y = 34 $\therefore \tau = 0$

At position B, y = 14 $\therefore \tau = 10.4\ Nmm^{-2}(MPa)$

In the lower section:

Consider a position in the lower section, again vertical distance y from the centroid, G, as shown in Figure 7.16(b).

At position B, there is a step change in the shear stress given by the ratio of the section widths at this point. Thus,

At position B $\qquad \tau = \dfrac{10.4.80}{40} = 20.8\ Nmm^{-2}(MPa)$

At position G, i.e. the neutral axis, we can use the discrete formula for the shear stress. In this case, to simplify calculation, the relevant area can be regarded as the area below the neutral axis. Thus,

At position G $$\tau = \frac{S}{I\,z} A\,\bar{y} = \frac{S}{2.31x10^6.40}.(46.40).23$$
$$= 22.91\ Nmm^{-2}\,(MPa)$$

At position C $\tau = 0$ i.e. a free surface

A sketch of the variation of the shear stress down the vertical centre line is now given in Figure 7.17.

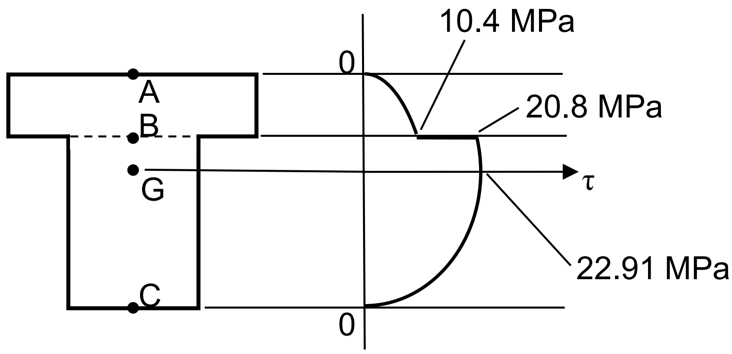


Figure 7.17

7.5.2 Shear Centre of thin-walled semi-circular cross-section

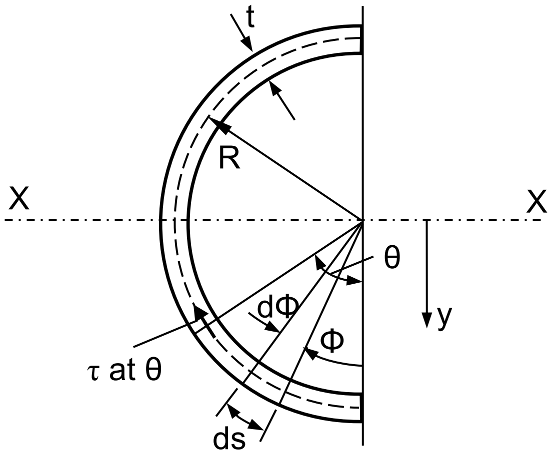


Figure 7.18

For the thin-walled semi-circular cross-section shown in Figure 7.18, determine the position of the shear centre (assume bending about the axis of symmetry X-X)

Shear stress distribution:

To solve this problem it is necessary to change from a rectangular co-ordinate system (x-y) to a polar co-ordinate system (r-θ). Referring to Figure 7.18 and using the integral form of the shear stress formula, we obtain a general expression for the shear stress distribution parallel to the wall of the section. Thus,

$$\tau = \frac{S}{I\,z}\int_A y\,dA \qquad \text{with} \quad y = R\cos\Phi$$

$$dA = ds.t = Rtd\Phi$$

$$z = t$$

giving,

$$\tau = \frac{S}{I\,z}\int_0^{\theta} R\cos\phi . R\,t\,d\phi$$

$$= \frac{SR^2}{I}\int_0^{\theta}\cos\phi\,d\phi$$

$$\therefore \tau = \underline{\frac{SR^2\sin\theta}{I}} \qquad [7]$$

Now,

$$I = \int_A y^2 dA = \int_0^{\pi}(R\cos\theta)^2 . R\,d\theta . t$$

$$= \int_0^{\pi} R^3\,t\cos^2\theta\,d\theta$$

$$= \int_0^{\pi}\frac{R^3 t}{2}(1+\cos 2\theta)d\theta$$

$$= \frac{R^3 t}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$\therefore I = \underline{\frac{\pi R^3 t}{2}} \qquad [8]$$

From [7] and [8],

$$\tau = \frac{SR^2\sin\theta . 2}{\pi R^3 t}$$

$$\underline{\therefore \tau = \frac{2\,S\sin\theta}{\pi\,R\,t}}$$

Shear centre:

The twisting moment (torque) associated with the above shear stress distribution for the whole cross-section is found by taking moments about O,

Torque

$$= \int_0^{\pi}\tau.(Rd\theta)\,t\,.R = \frac{R^2 t.2S}{\pi Rt}\int_0^{\pi}\sin\theta\,d\theta$$

$$= \frac{2SR}{\pi}\left[-\cos\theta\right]_0^{\pi} = \underline{\frac{4SR}{\pi}}$$

To counteract this twisting moment, as shown in Figure 7.19, the shear force, S, must be applied at the shear centre, a distance e, given by,

$$S(e+R) = \frac{4SR}{\pi}$$

$$\therefore e = \frac{4R}{\pi} - R = \underline{0.273R}$$

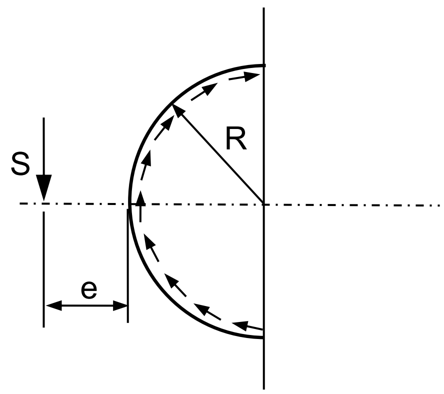


Figure 7.19