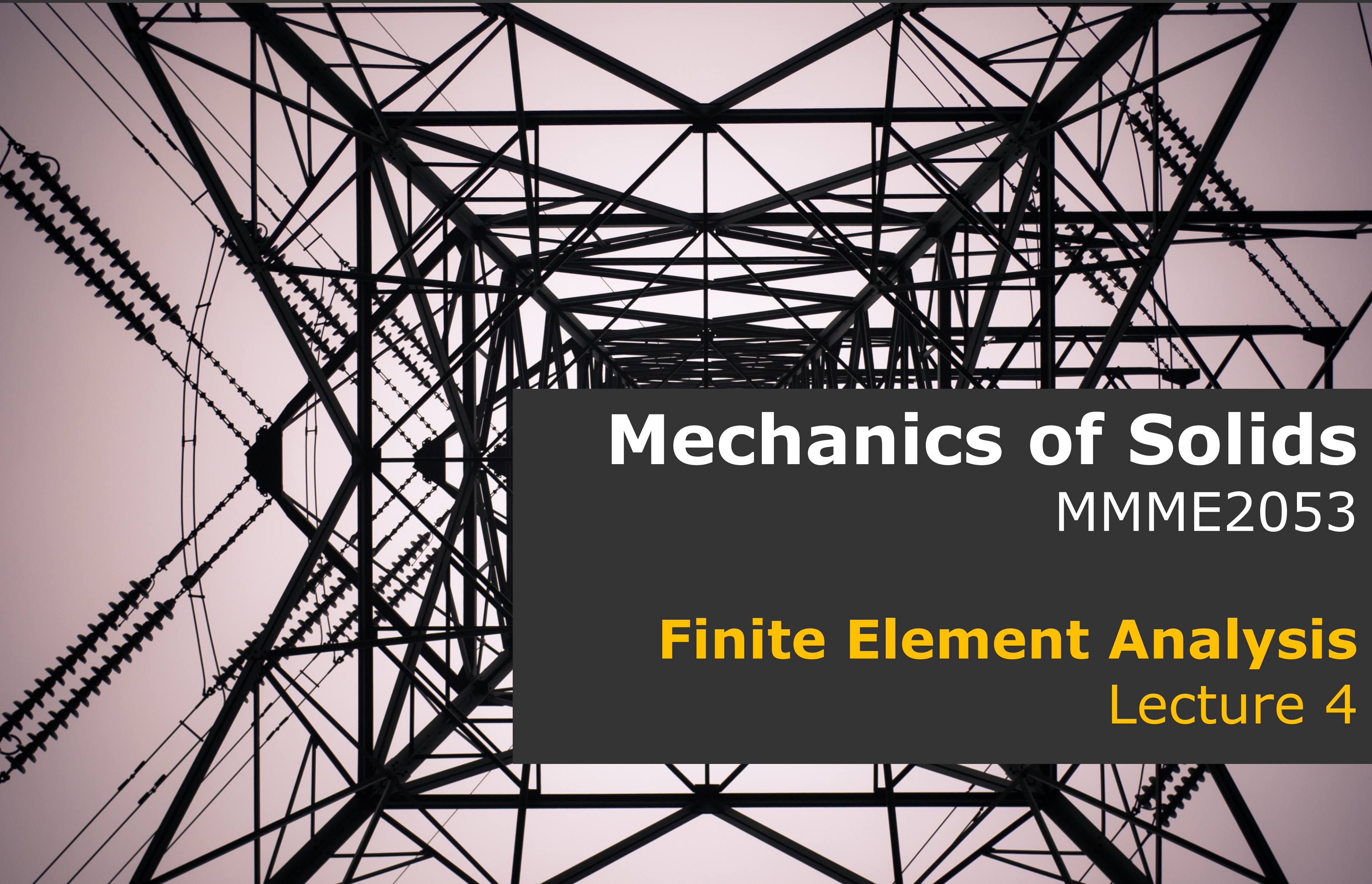


Mechanics of Solids
MMME2053

Finite Element Analysis
Lecture 4

Learning Objectives

1. Recognise that FEA is a useful technique to aid the solution of many Structural Mechanics problems
2. Understand how 1D elements and the matrix method can be used to analyse uniaxial bars
3. Apply theory for 1D elements and the matrix method to an assembly of bars
4. Understand the derivation of the global stiffness matrix of a truss element

Learning Objectives

5. Understand how shape functions allow us to determine displacement (and stresses and strains within an element)

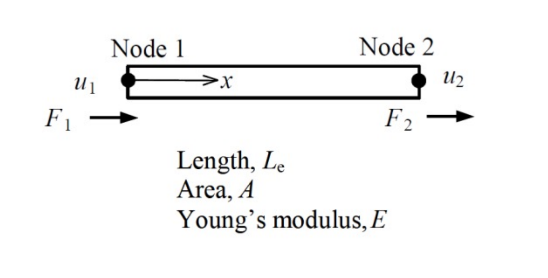
6. Understand how 2D approximations can be used to simplify the modelling of 3D problems

7. Understand how (simple 2D) continuum elements allow analysis of structures

Shape Functions (1)

- For our 1D bar, the displacement may be represented as a linear polynomial

$$u = \alpha_1 + \alpha_2 x$$



- α_1 and α_2 are constants which may be determined from the nodal displacements and the geometry of the element

- At node 1, x = 0 so, $u = u_1 = \alpha_1$
- At node 2, x = L so, $u = u_2 = \alpha_1 + \alpha_2 L$

Shape Functions (2)

- From this $\alpha_1 = u_1$
- And $\alpha_2 = \dfrac{u_2 - u_1}{L}$
- So the resultant displacement over the element is:

$$u(x) = \alpha_1 + \alpha_2 x$$

$$u(x) = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2$$

Shape Functions (2)

- Or in matrix form:

$$u(x) = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

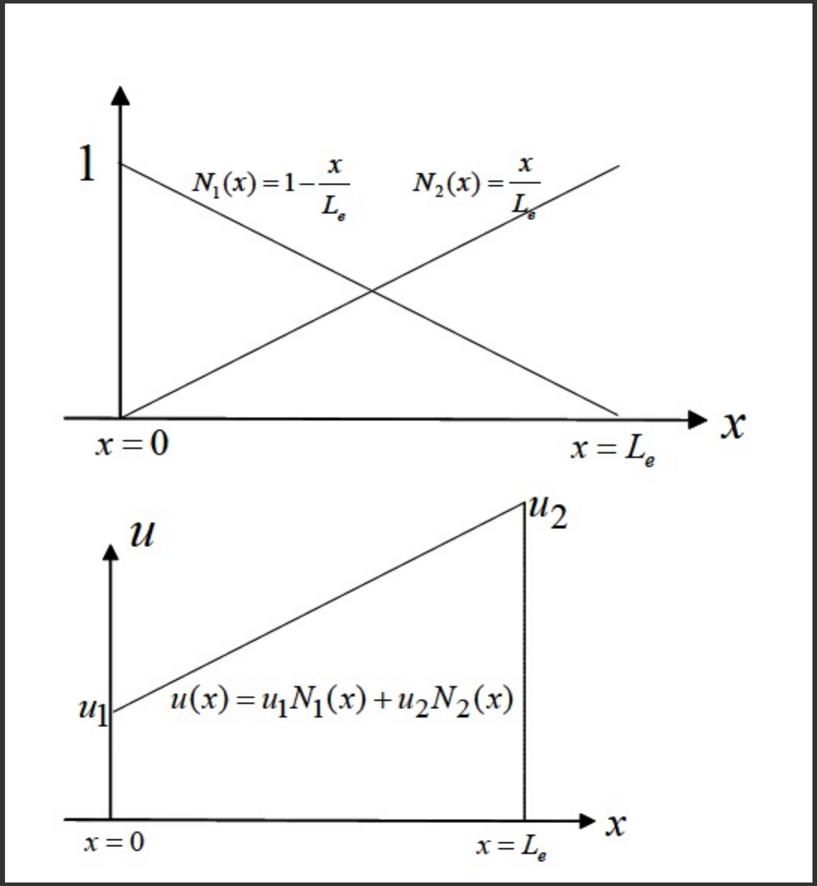
$$u(x) = [N_1(x) \quad N_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u(x) = [N]\{u\}$$

where [*N*] (or $N_1(x)$ and $N_2(x)$) are the **shape functions** of the element, which specify the variation in displacement within the element, here it is linear

Shape Functions (3)

- We can combine these shape functions to define the variation in displacement across the element as shown
- The principle is the same for 2D and 3D elements



Shape Functions (4)

- The shape functions can also be used to determine the strain in the element

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[N]\{u\}$$

$$\varepsilon_x = \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\varepsilon_x = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\varepsilon_x = [B]\{u\}$$

Shape Functions (5)

- And then the stress (for a uniaxial bar):

$$\sigma = \varepsilon E = [E][B]\{u\}$$

$$\sigma = E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- In this case, [*E*] is just a single value. For more complicated 2D and 3D elements, more elements are required in [*E*] but the principle is the same

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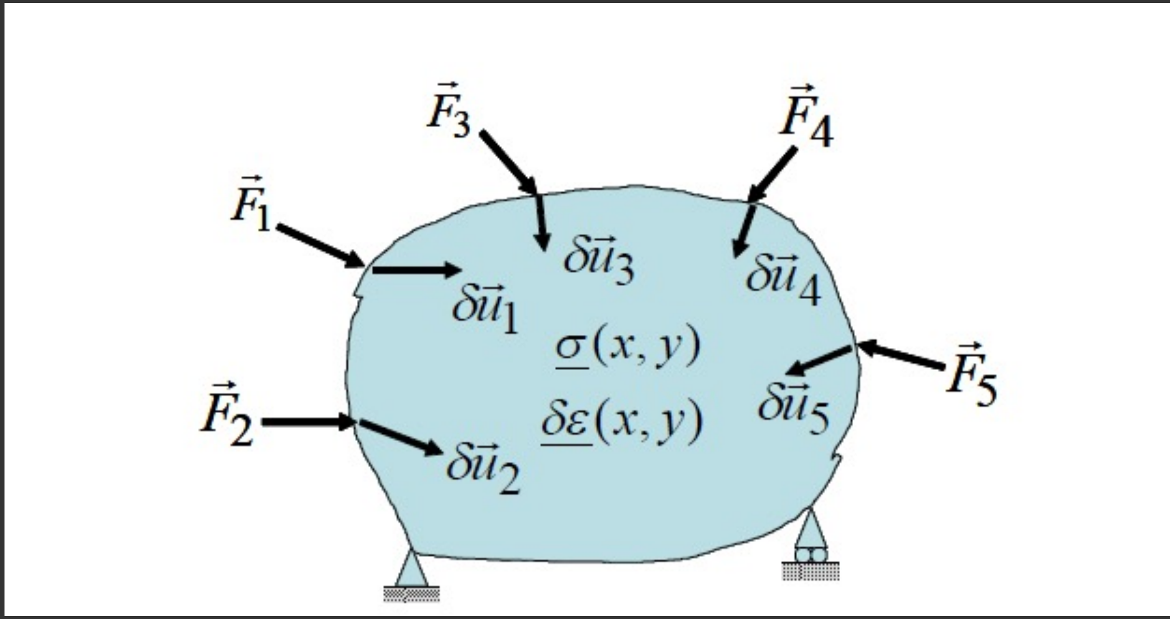
The Principle of Virtual Work (1)

- The principle of virtual work is fundamental to structural finite element (FE) analysis as it permits the development of expressions for the stiffness matrices of a range of different structural element types.
- This permits computational modelling of complex geometries to give approximate solutions for displacement, stress and strain distributions.

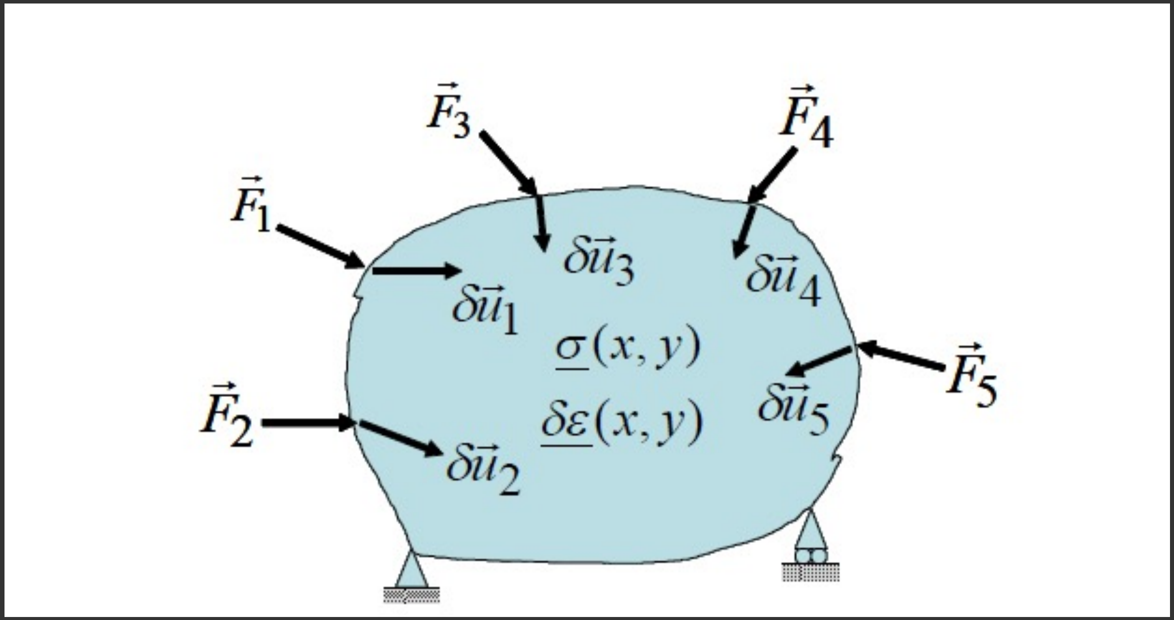
The Principle of Virtual Work (2)

- If we consider a 2D body in equilibrium under a set of vector forces, leading to a set of vector displacements and also to an internal stress and strain distribution:



The Principle of Virtual Work (2)

- The Principle of Virtual Work states that the work done by a set of forces moving through a set of small, compatible displacements is zero



$$\delta W = \delta W_{int} - \delta W_{ext} = 0$$

The Principle of Virtual Work (3)

- Internal Virtual Work:

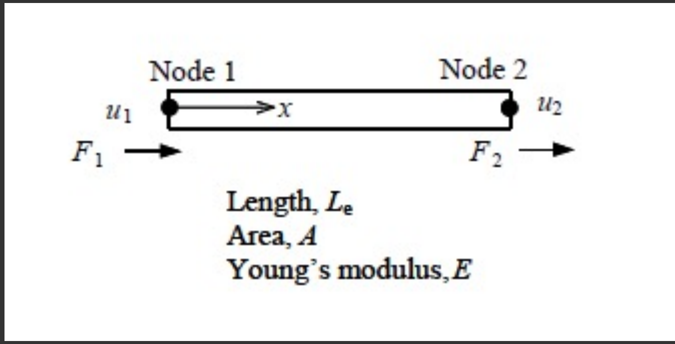
$$\delta W_{int} = \int_V \sigma \delta \varepsilon^T dV$$

- External Virtual Work:

$$\delta W_{ext} = \int_s \delta u F dS$$

The Principle of Virtual Work (4)

- Returning to our 1D element



- Internal Virtual Work:

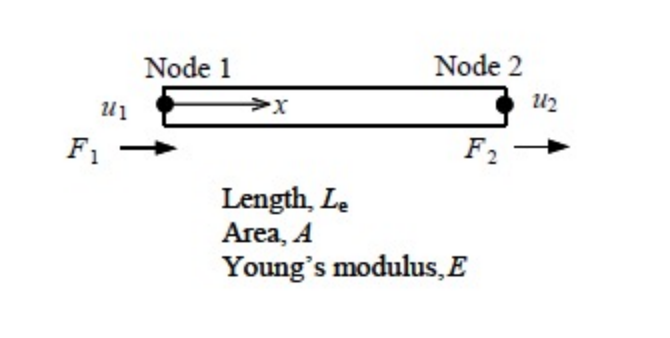
$$\delta W_{int} = \int_0^{L_e} \delta\varepsilon^T \sigma A dx$$

$$\delta W_{int} = A \int_0^{L_e} \{\delta u\}^T \{B\}^T E \{B\}\{u\} dx$$

$$\delta W_{int} = EA\{\delta u\}^T \int_0^{L_e} \{B\}^T \{B\} dx \{u\}$$

The Principle of Virtual Work (5)

- Returning to our 1D element



- External Virtual Work:

$$\delta W_{ext} = \delta u_1 F_1 + \delta u_2 F_2 = \{\delta u\}^T \{F\}$$

The Principle of Virtual Work (6)

- The Principle of Virtual Work states:

$$\delta W = \delta W_{int} - \delta W_{ext} = 0$$

$$EA\{\delta u\}^T \int_0^{L_e} \{B\}^T \{B\} dx \{u\} = \{\delta u\}^T \{F\}$$

$$EA \int_0^{L_e} \{B\}^T \{B\} dx \{u\} = \{F\}$$

The Principle of Virtual Work (7)

- Consider:

$$EA\int_0^{L_e}\{B\}^T\{B\}dx = EA\int_0^{L_e}\begin{Bmatrix} -\dfrac{1}{L_e} \\ \dfrac{1}{L_e} \end{Bmatrix}\begin{Bmatrix} -\dfrac{1}{L_e} & \dfrac{1}{L_e} \end{Bmatrix}dx$$

$$= \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [K]$$

- So the stiffness equation becomes

$$[K]\{u\} = \{F\} \qquad \text{as before}$$

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