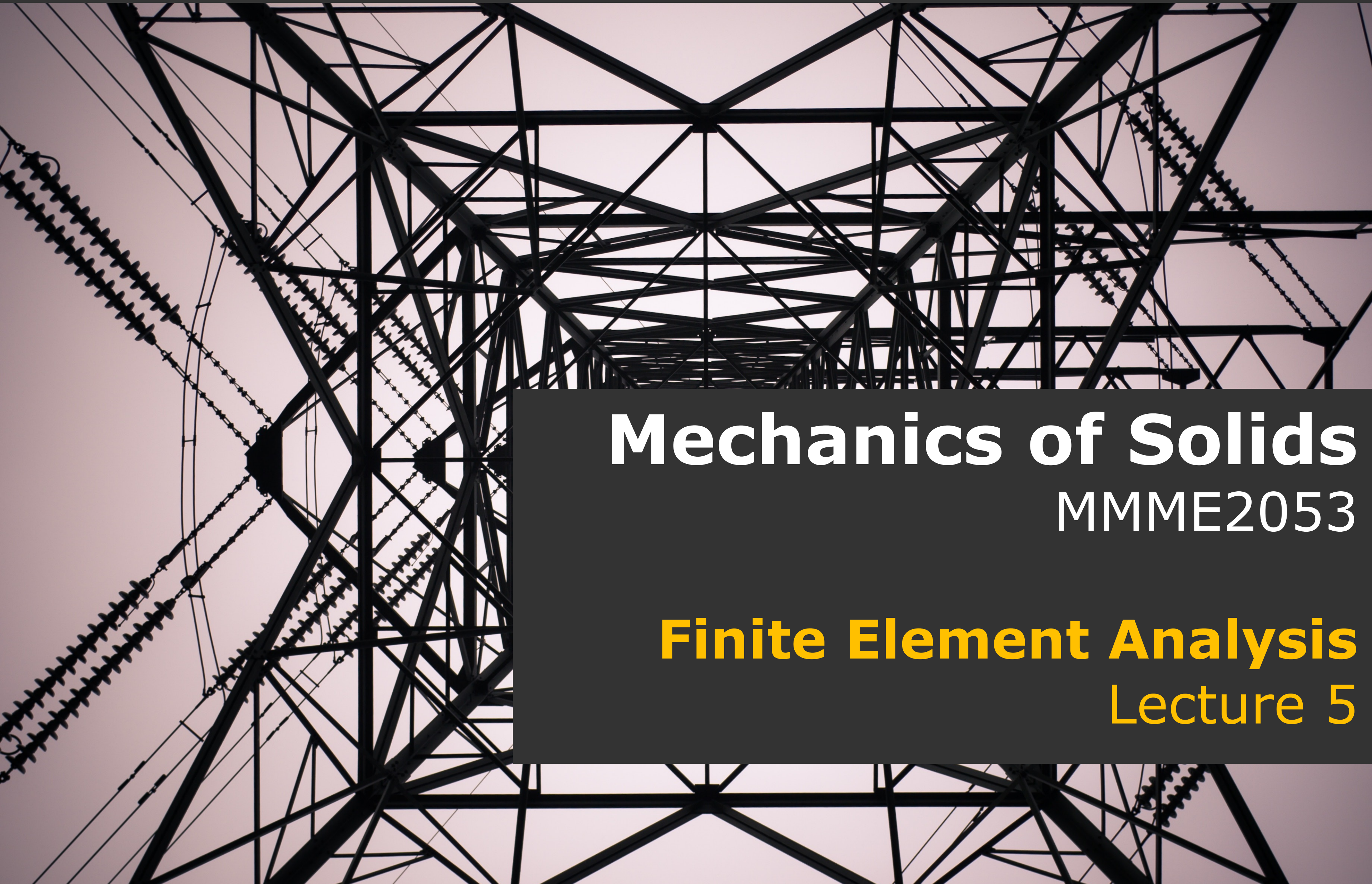


Mechanics of Solids
MMME2053

Finite Element Analysis
Lecture 5

Learning Objectives

5. Understand how shape functions allow us to determine displacement (and stresses and strains within an element)

6. Understand how 2D approximations can be used to simplify the modelling of 3D problems

7. Understand how (simple 2D) continuum elements allow analysis of structures

2D Approximations

- Although problems are 3D, we can make some assumptions and use a 2D approach for some problems
- Useful assumptions include the **plane stress, plane strain** and **axisymmetric** assumptions

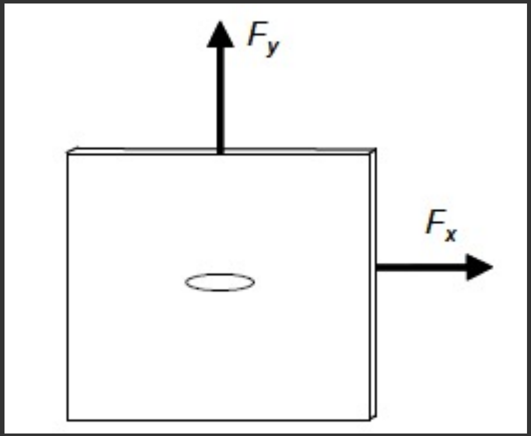
2D Approximations

- **Plane Stress** Approximation

- Consider a thin plate which is only loaded in the in-plane directions

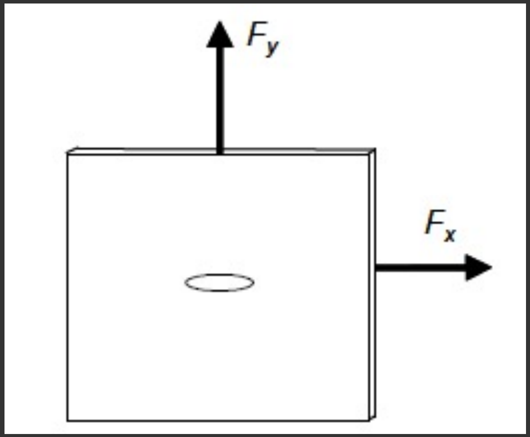
- The normal stress σ_z must be zero on the front and back faces

- Because the plate is thin, then we can assume that $\sigma_z \approx 0$ throughout the thickness



2D Approximations

- **Plane Stress** Approximation



- The only non-zero components of stress are σ_x, σ_y and τ_{xy} and we can determine all of the strain components, i.e. ε_x, ε_y, ε_z and γ_{xy}, from these stress components using Hooke's law (for elastic behaviour)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)$$

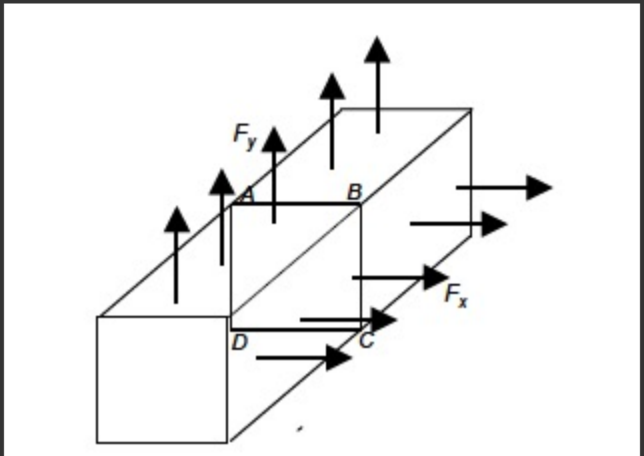
$$\varepsilon_x = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

2D Approximations

- **Plane Strain** Approximation

- Consider a very thick plate or long member of regular cross-section, again only loaded in the in-plane directions



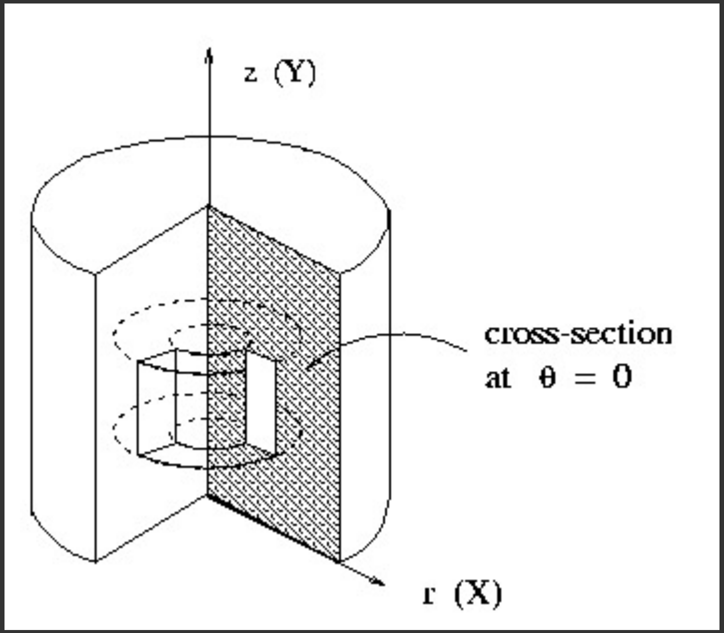
- A plane ABCD, remote from the ends experiences negligible strain in the z-direction $\varepsilon_z \approx 0$

- We can determine the z-direction stresses from the x and y-direction normal stresses

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

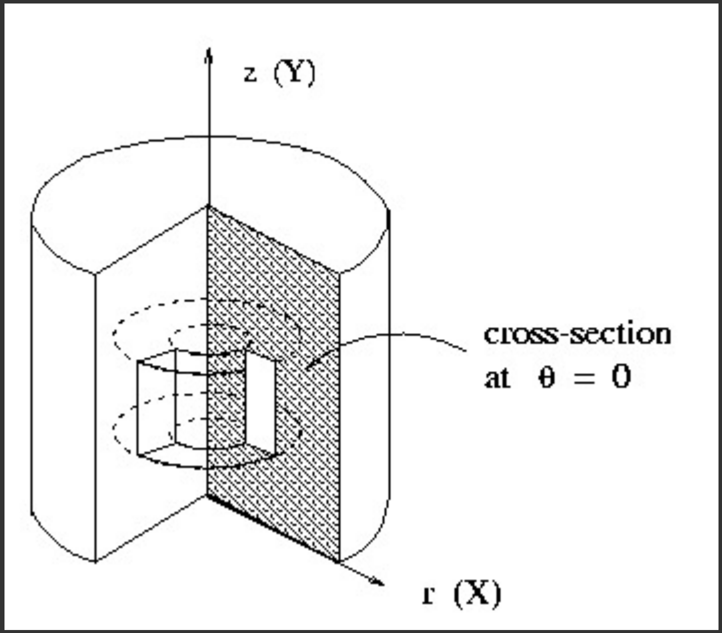
2D Approximations

- **Axisymmetric** Approximation
- Used to represent cases with geometry and loading that is rotationally symmetric (r, z, θ coordinates)
- Because of symmetry about the z axis, the stresses are independent of the θ coordinate



2D Approximations

- **Axisymmetric** Approximation
- All derivatives with respect to θ vanish and the displacement component in the θ direction, the shear strains $\gamma_{r\theta}$ and $\gamma_{\theta z}$ and the shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ are all zero



2D Approximations

- Plane Stress:

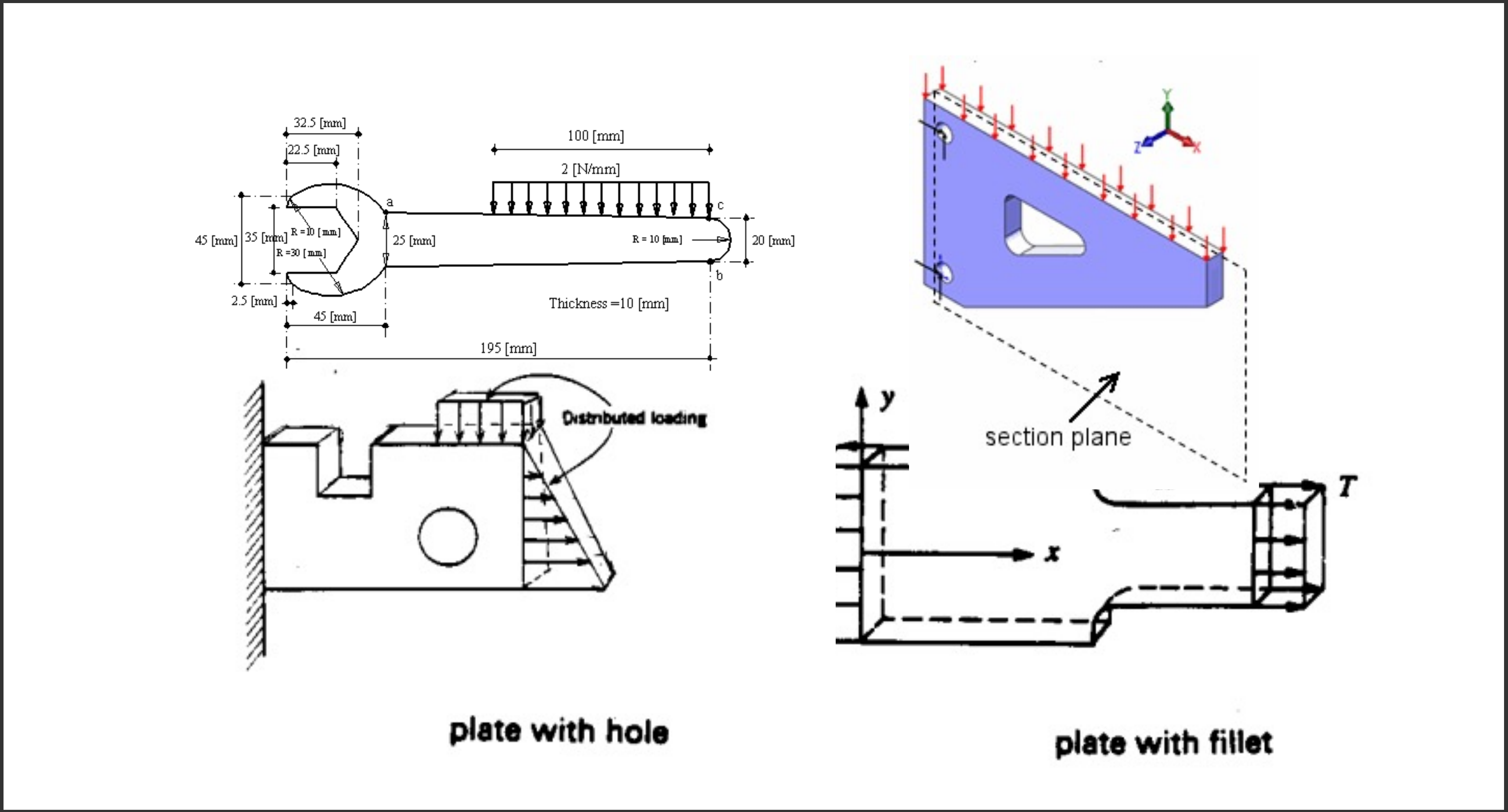
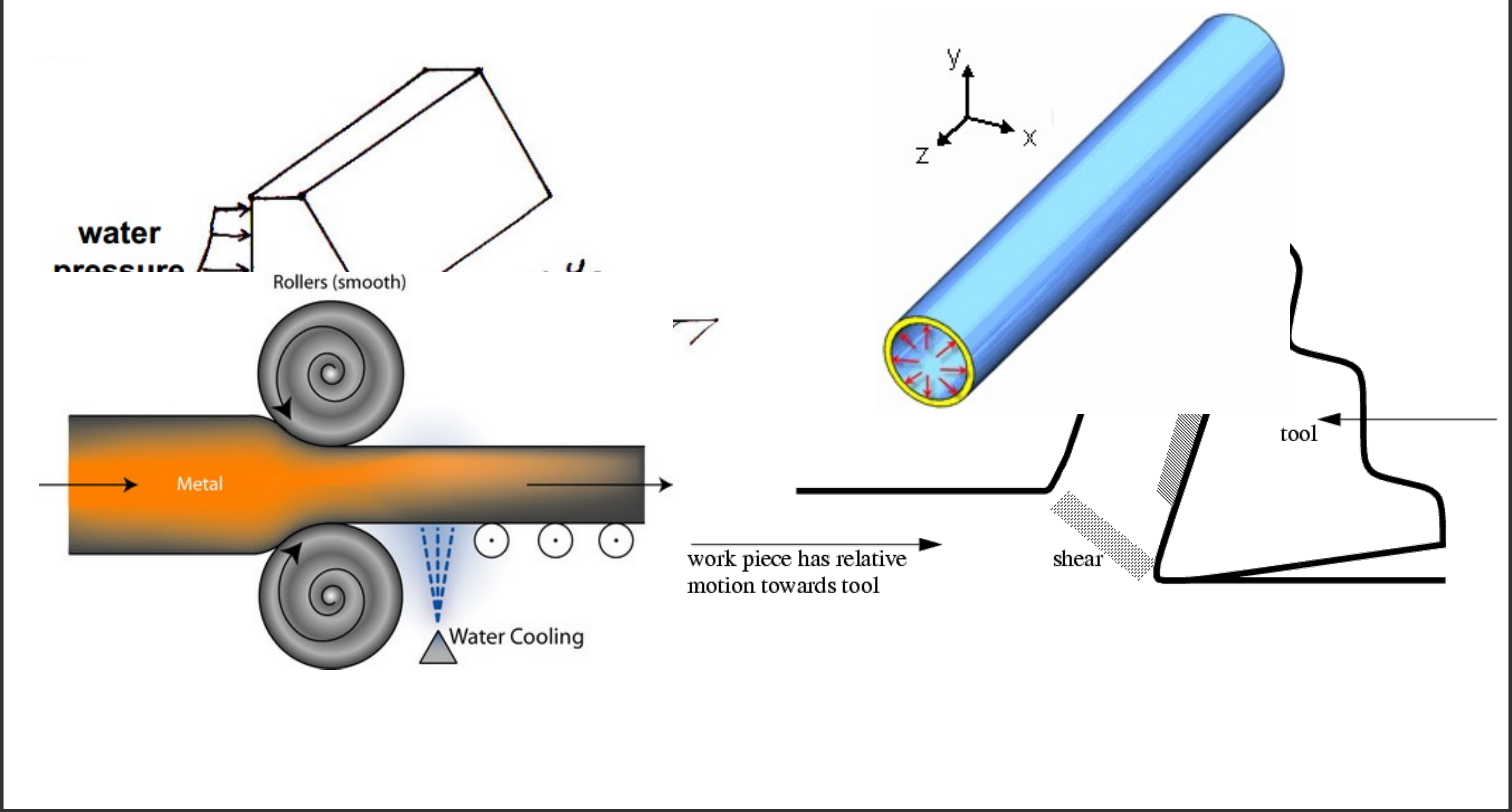


plate with hole

plate with fillet

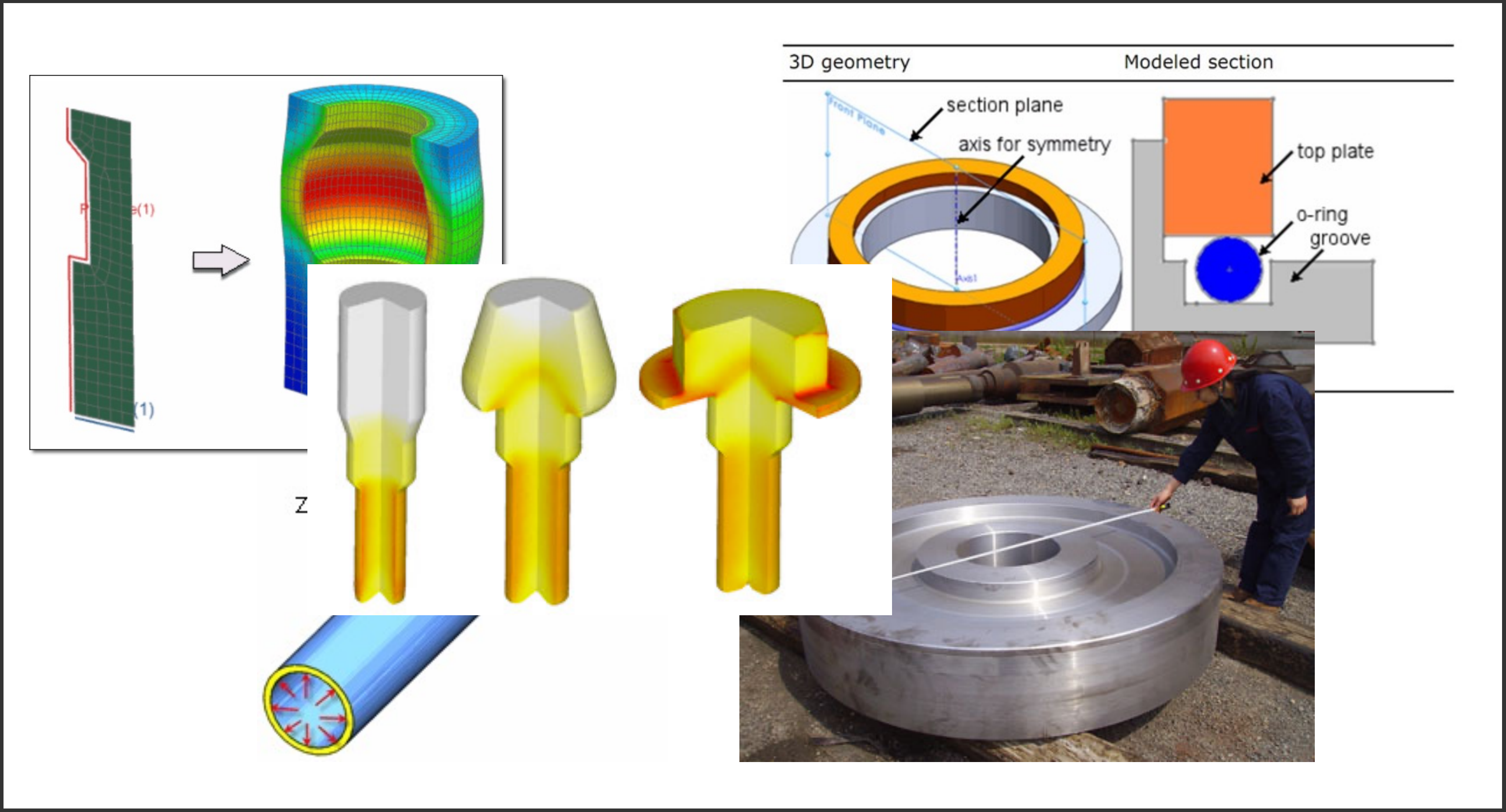
2D Approximations

- Plane Stress:



2D Approximations

- Axisymmetric:



Objectives

5. Understand how shape functions allow us to determine displacement (and stresses and strains within an element)

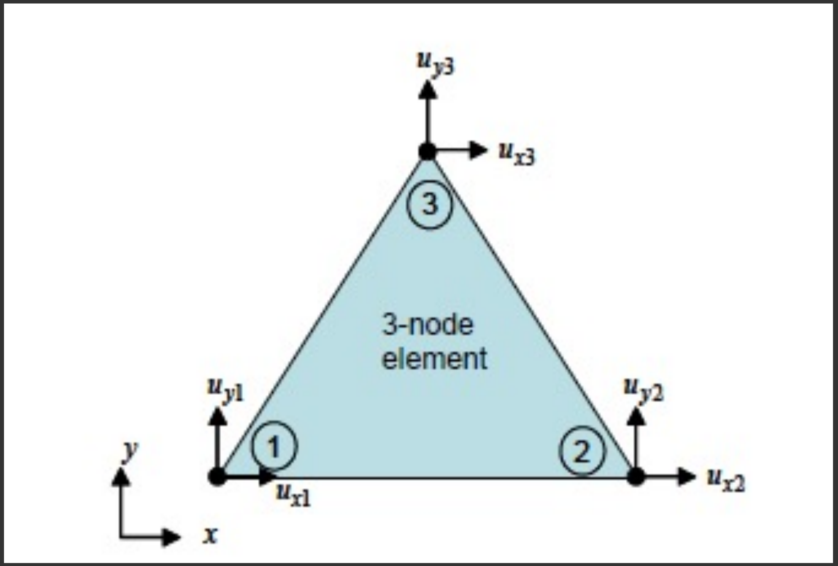
6. Understand how 2D approximations can be used to simplify the modelling of 3D problems

7. Understand how (simple 2D) continuum elements allow analysis of structures

Constant Strain Triangle (1)

- The simplest 2D element
- A 3-noded, linear triangular element, 2 DOF per node
- Linear variation of displacements within the element:

$$u_x(x, y) = C_1 + C_2 x + C_3 y$$
$$u_y(x, y) = C_4 + C_5 x + C_6 y$$



Constant Strain Triangle (2)

- The constants C_1 – C_6 are related to the nodal coordinates and the values of the displacements by:

$$\begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix}$$

$$\{u\} = [A]\{C\}$$

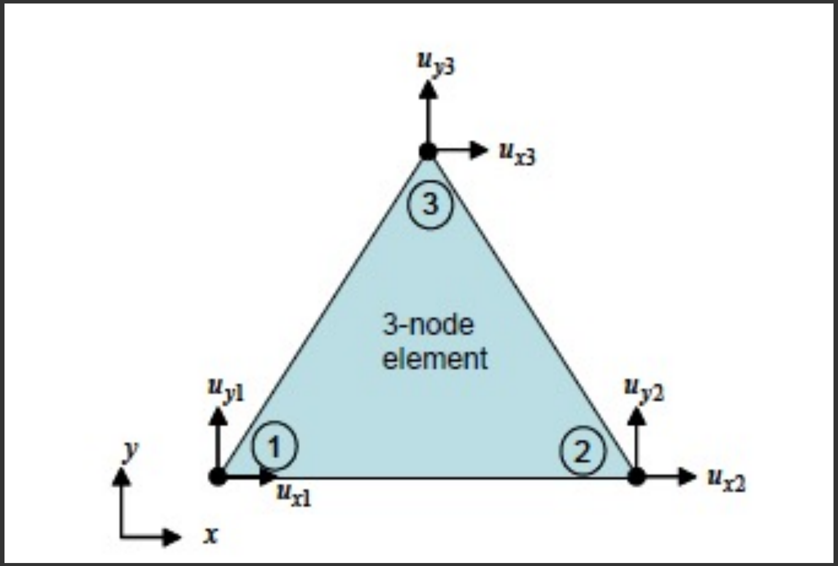
Constant Strain Triangle (3)

- The strain field is calculated by:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = C_2$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = C_6$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = C_3 + C_5$$



- All constant values – no variation within the element – hence the name

Constant Strain Triangle (4)

- Which in matrix form is:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix}$$

$$\{\varepsilon\} = [X]\{C\}$$

Constant Strain Triangle (5)

- From before:

$$\{C\} = [A]^{-1}\{u\}$$

- So:

$$\{\varepsilon\} = [X][A]^{-1}\{u\} = [B]\{u\}$$

Constant Strain Triangle (6)

- If you work through the maths:

$$[B] = \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_3 & y_1 - y_2 \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix}$$

- These are all known quantities as they are all a function of the coordinates of the nodes

Constant Strain Triangle (7)

- To relate stresses to strains, we require a matrix of elastic constants (Hooke's law)
- For plane stress cases (as mentioned before) this is (in matrix form):

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Constant Strain Triangle (8)

- If we rearrange this:

$$\{\sigma\} = [D]\{\varepsilon\}$$

- Where D (for plane stress) is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Constant Strain Triangle (9)

- Substituting for the strains

$$\{\sigma\} = [D][B]\{u\}$$

- Which allows us to determine the element stresses from the element displacements

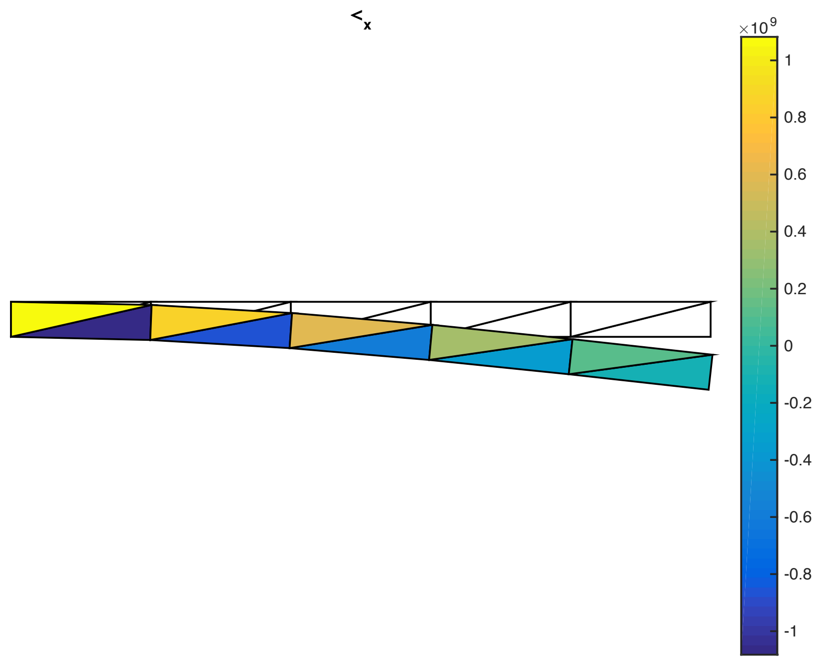
CST Example

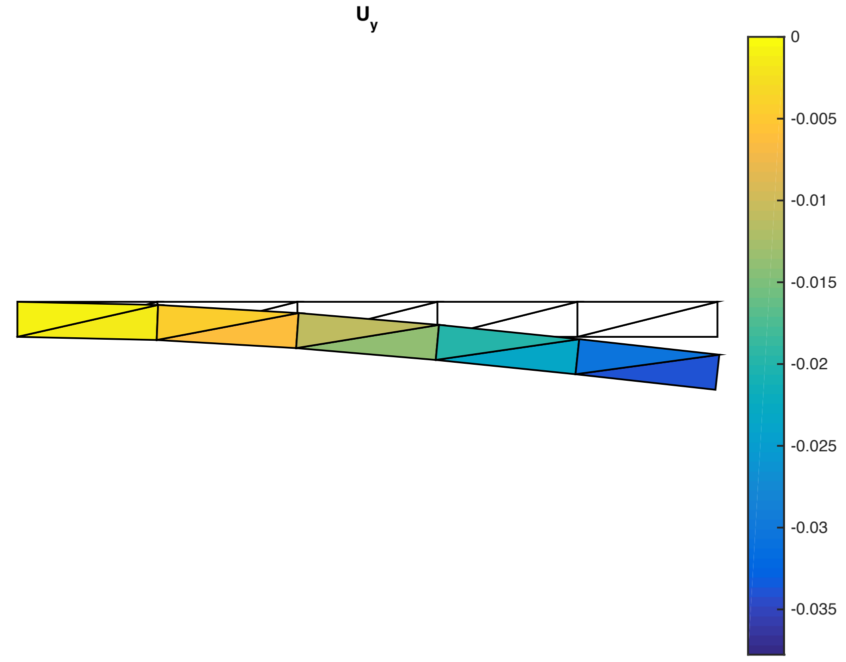
- A steel cantilever beam of length 0.5m width 0.01m and depth 0.025m, is subjected to a point load at the end of 5kN, determine the maximum stress in the beam and the maximum deflection of the tip of the cantilever.

E = 200 GPa

CST Example

N_{el} = 10 (1 through thickness)



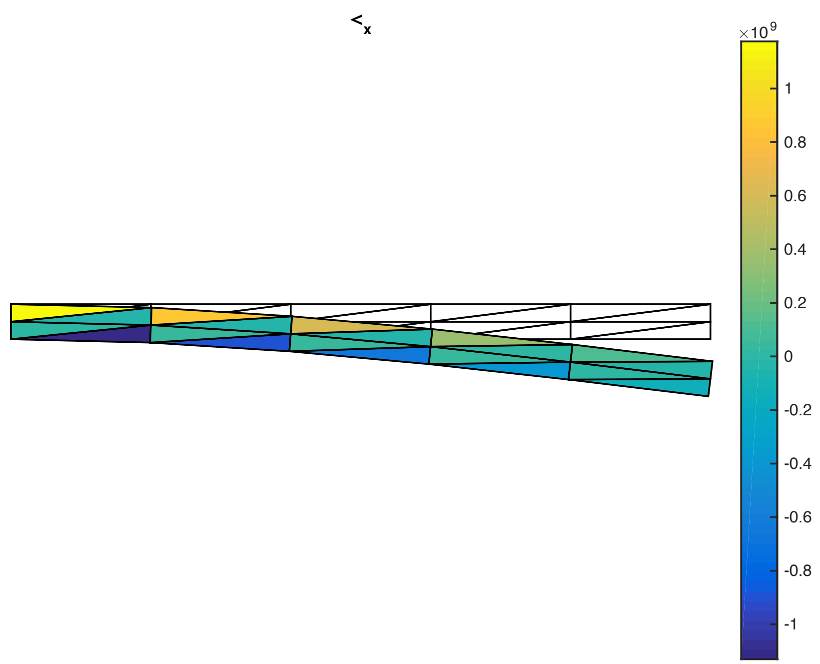


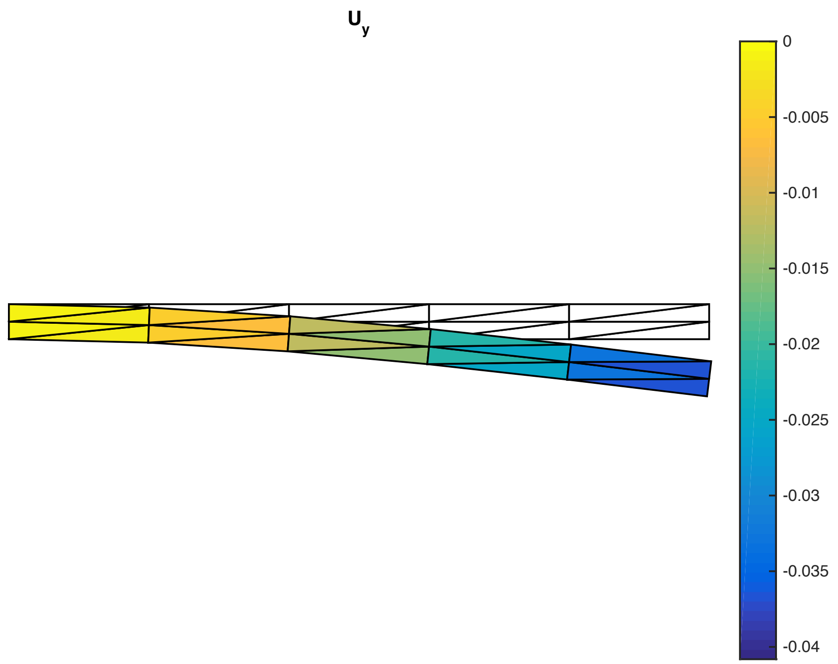
σ_{x_max} = 1082 MPa

u_{y_min} = -0.0378 m

CST Example

N_{el} = 20 (2 through thickness)



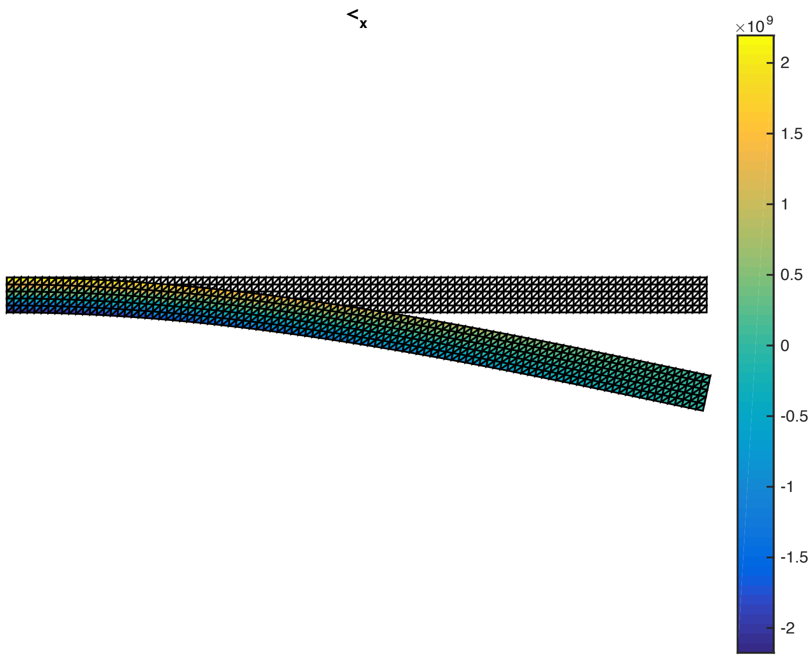


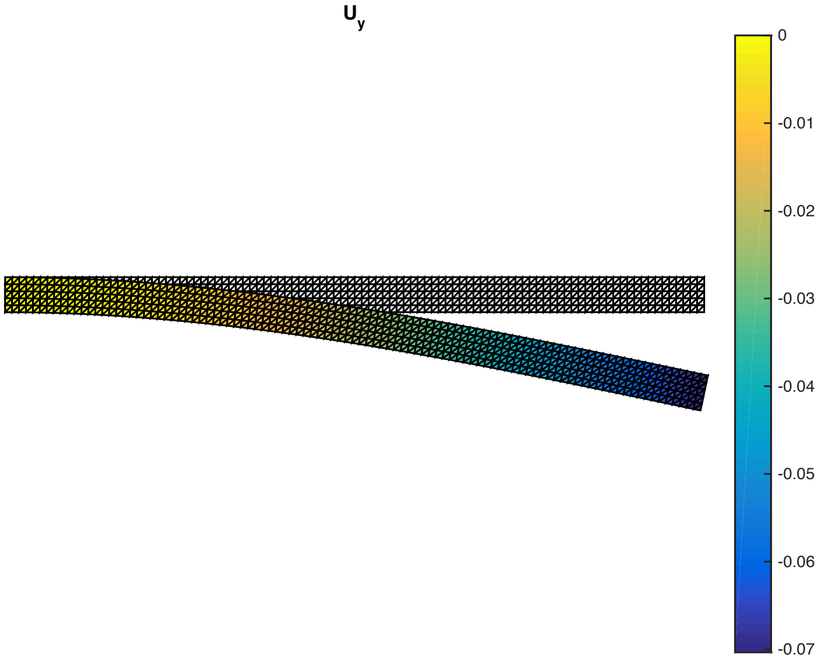
σ_{x_max} = 1176 MPa

u_{y_min} = -0.0408 m

CST Example

N_{el} = 1000 (5 through thickness)





σ_{x_max} = 2195 MPa

u_{y_min} = -0.0703 m

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