The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Mechanics of Solids
MMME2053

Thick Cylinders
Lecture 2

Analysis of Thick Cylinders

- Thick cylinder problems are *statically indeterminate*
  - In order to obtain a solution, it is necessary to consider:
    - Equilibrium,
    - Compatibility
    - Material behaviour (stress-strain relationship)

Assumptions

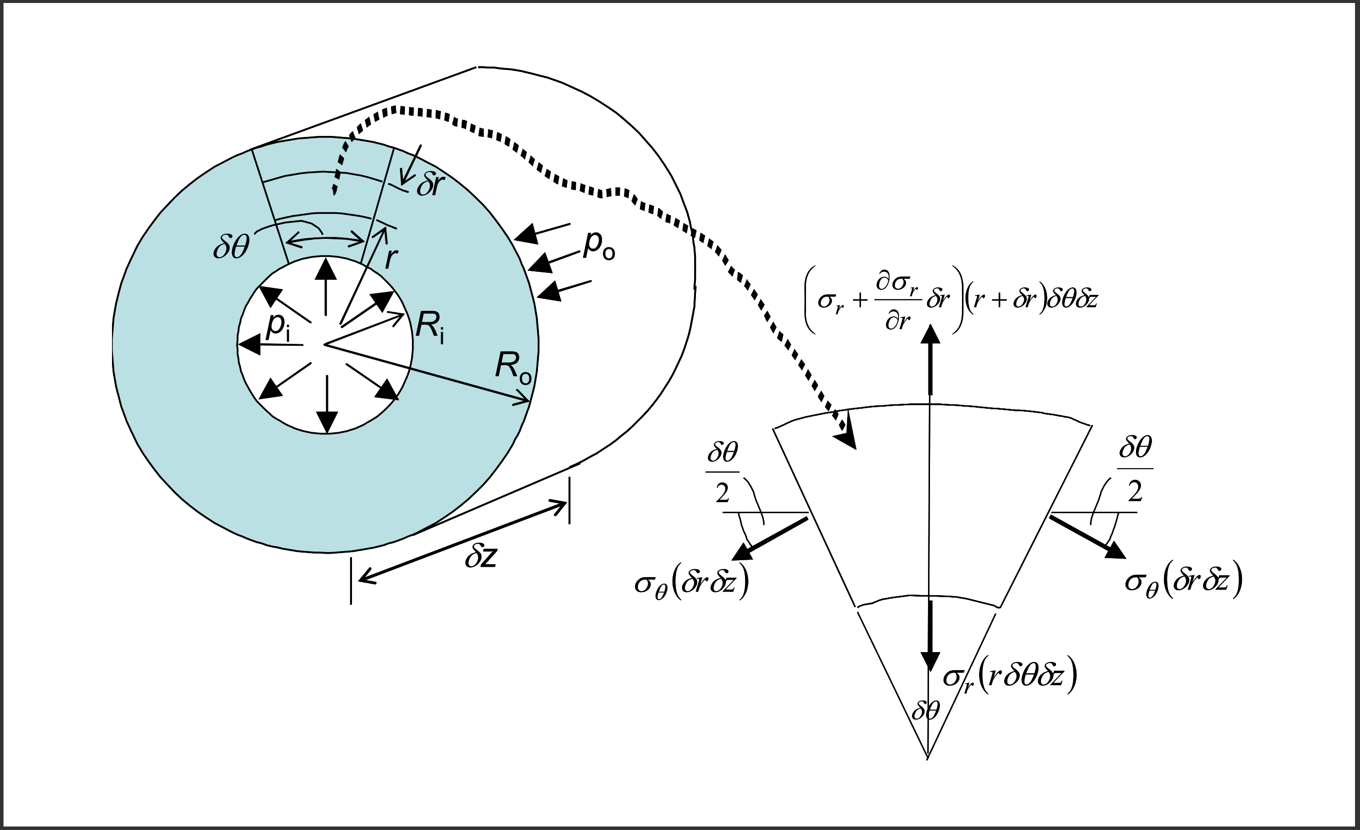
i. Plane transverse sections remain plane (this is true remote from the ends)

ii. Deformations are small

iii. The material is linear elastic, homogenous and isotropic.

Derivation of Lame's Equations

- **Equilibrium**



r, *θ*, *z* coordinate system

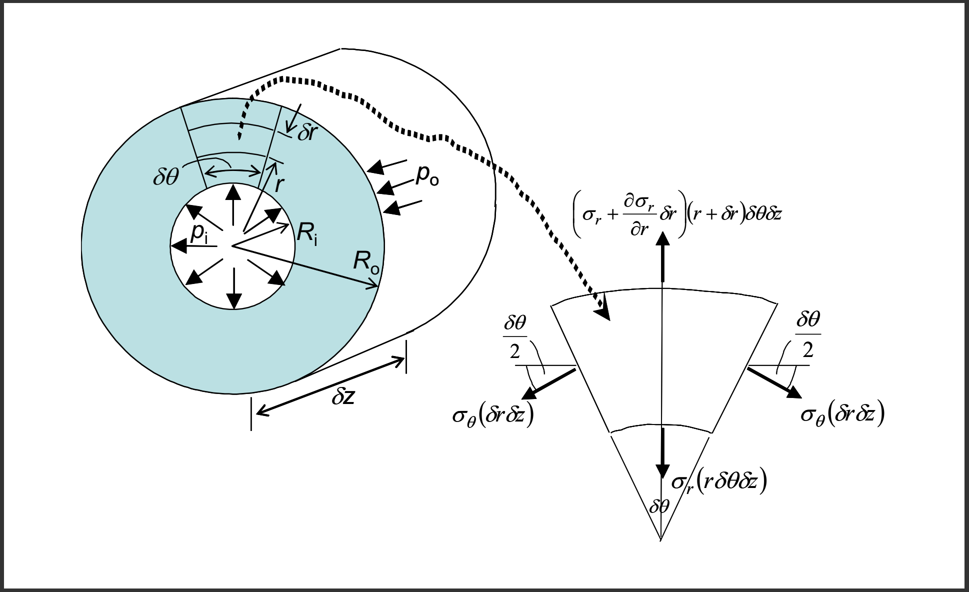
σ_r = radial stress

σ_θ = hoop stress

σ_z =axial stress

Derivation of Lame's Equations

- **Equilibrium**



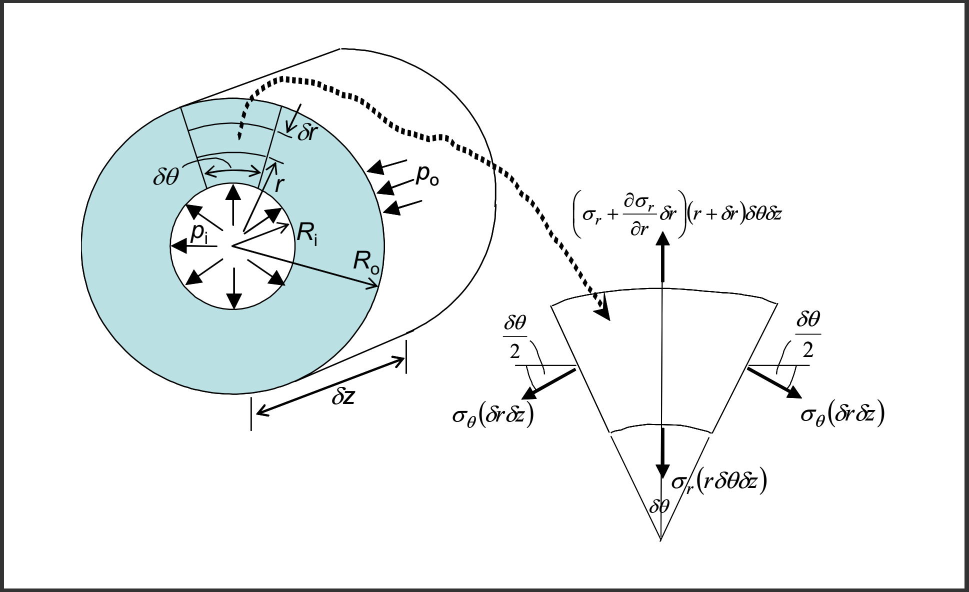
$$\left(\sigma_r + \frac{d\sigma_r}{dr}\delta r\right)(r+\delta r)\delta\theta\delta z = \sigma_r(r\delta\theta\delta z) + 2\sigma_\theta(\delta r\delta z)\sin\left(\frac{\delta\theta}{2}\right) \qquad (1)$$

For small $\delta\theta$, $\sin\left(\frac{\delta\theta}{2}\right) \approx \frac{\delta\theta}{2}$ therefore:

$$\sigma_r(r+\delta r)\delta\theta + \frac{d\sigma_r}{dr}\delta r\,(r+\delta r)\delta\theta = \sigma_r r\delta\theta + \sigma_\theta\delta r\delta\theta$$

Derivation of Lame's Equations

- **Equilibrium**



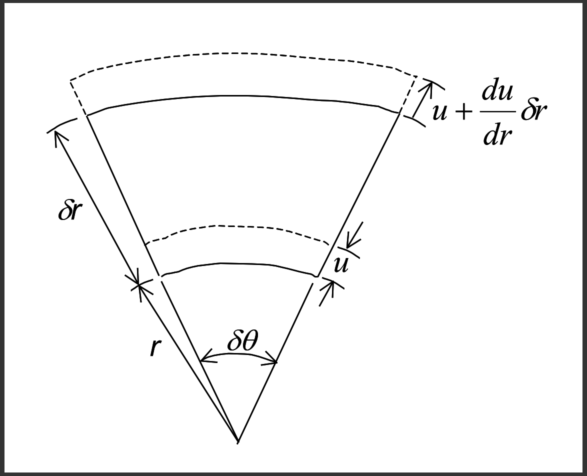
$$r\sigma_r + \sigma_r\delta r + r\frac{d\sigma_r}{dr}\delta r + \frac{d\sigma_r}{dr}\delta r^2 = \sigma_r r + \sigma_\theta \delta r \qquad (2)$$

As $\delta r \to 0, \frac{d\sigma_r}{dr}\delta r^2 \to 0$

$$\sigma_\theta - \sigma_r = r\frac{d\sigma_r}{dr} \qquad (3)$$

Derivation of Lame's Equations

- **Compatibility**



$$\varepsilon = \frac{\text{extension}}{\text{original length}}$$

$$\text{Hoop strain, } \varepsilon_\theta = \frac{(r+u)\delta\theta - r\delta\theta}{r\delta\theta} = \frac{u}{r} \qquad (4)$$

$$\text{Radial strain, } \varepsilon_r = \frac{\left(u + \frac{du}{dr}\delta r\right) - u}{\delta r} = \frac{du}{dr} \qquad (5)$$

$$\text{Axial strain, } \varepsilon_z = \text{constant} \qquad (6)$$

Derivation of Lame's Equations

- **Material Behaviour**

- Generalised Hooke's Law

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_z)) \quad (7)$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu(\sigma_\theta + \sigma_z)) \quad (8)$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_r + \sigma_\theta)) \quad (9)$$

(assuming linear elastic, isotropic material behaviour)

Derivation of Lame’s Equations

- Equations (3) to (9) have seven unknowns,

- i.e. u, σ_θ, σ_r, σ_z, ε_θ, ε_r, ε_z, which are all functions of r, p_o, p_i, R_o, R_i, ν and E.

- Substituting $u = r\varepsilon_\theta$ from Eq. (4) into Eq. (5) gives

$$\varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta)$$

$$\varepsilon_r = \varepsilon_\theta + r\frac{d\varepsilon_\theta}{dr} \qquad \text{(a)}$$

- Using Eq. (7) and Eq. (8) in Eq. (a) gives

$$\frac{1}{E}\left(\sigma_r - \nu(\sigma_\theta + \sigma_z)\right) = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_z)) + \frac{\mathrm{r}}{\mathrm{E}}\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_z}{dr}\right)$$

$$(1+\nu)\sigma_r = (1+\nu)\sigma_\theta + \mathrm{r}\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr} - r\nu\frac{d\sigma_z}{dr} \qquad \text{(b)}$$

Derivation of Lame’s Equations

- Using Eq. (6) $\frac{\mathrm{d}\varepsilon_z}{dr} = 0$ then Eq. (9) gives

$$0 = \frac{d\sigma_z}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_\theta}{dr}$$

$$\frac{d\sigma_z}{dr} = \nu\frac{d\sigma_r}{dr} + \nu\frac{d\sigma_\theta}{dr} \qquad \text{(c)}$$

- Substituting (c) in (b)

$$(1+\nu)\sigma_r = (1+\nu)\sigma_\theta + \mathrm{r}\frac{d\sigma_\theta}{dr} - \mathrm{r}v\frac{d\sigma_r}{dr} - r\nu^2\frac{d\sigma_r}{dr} + r\nu^2\frac{d\sigma_\theta}{dr}$$

$$(1+\nu)\sigma_r = (1+\nu)\sigma_\theta + \mathrm{r}(1-\nu^2)\frac{d\sigma_\theta}{dr} - r\nu(1+\nu)\frac{d\sigma_r}{dr}$$

$$\sigma_\theta - \sigma_r = rv\frac{d\sigma_r}{dr} - (1-\nu)\frac{d\sigma_\theta}{dr} \qquad \text{(d)}$$

Derivation of Lame's Equations

- Substituting the right hand side from Eq. (d) into Eq. (3) gives:

$$r v \frac{d\sigma_r}{dr} - (1 - v)\frac{d\sigma_\theta}{dr} = r\frac{d\sigma_r}{dr}$$

$$r(1 - v)\left[\frac{d\sigma_r}{dr} + \frac{d\sigma_\theta}{dr}\right] = 0$$

$$\frac{d}{dr}(\sigma_r + \sigma_\theta) = 0$$

- Integration leads to:

$$\sigma_r + \sigma_\theta = 2A \qquad \text{(e)}$$

- Where A is a constant of integration

Derivation of Lame's Equations

- Again using Eq. (3) and substituting for σ_θ gives:

$$2\sigma_r = 2A - r\frac{d\sigma_r}{dr}$$

Rearranging

$$r\frac{d\sigma_r}{dr} + 2\sigma_r = 2A$$

Which is equivalent to

$$\frac{1}{r}\frac{d}{dr}(r^2\sigma_r) = 2A$$

$$r^2\sigma_r = \frac{2Ar^2}{2} - B$$

- Where B is another constant of integration, which leads to:

$$\boxed{\sigma_r = A - \frac{B}{r^2}}$$ and using Eq. (e) leads to $$\boxed{\sigma_\theta = A + \frac{B}{r^2}}$$

Analysis of Thick Cylinders

- The hoop and radial stresses at any point (radius, *r*) in the wall cross-section of a thick cylinder can be determined using *Lame's equations*:

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

- Where *A* and *B* are *Lame's constants* (constants of integration)

- Note that, since $\varepsilon_z = \mathrm{const}$ and$\sigma_r + \sigma_\theta = \mathrm{const}$, then Eq. (9) shows that $\sigma_z = \mathrm{const}$, i.e. it is independent of *r*. The value of σ_z can therefore be obtained from a consideration of axial equilibrium.

Learning Objectives

1. Appreciate the difference between the stress analysis of thin and thick cylinders (knowledge);
2. Understand the derivation of Lame's equations (comprehension);
3. Determine the stresses in a thick walled cylinder subjected internal and external pressure (application);
4. Determine the stresses caused by shrink fitting a cylinder onto another (application);
5. Be able to include 'inertia' effects into the thick cylinder equations to calculate the stresses in a rotating disc (application).