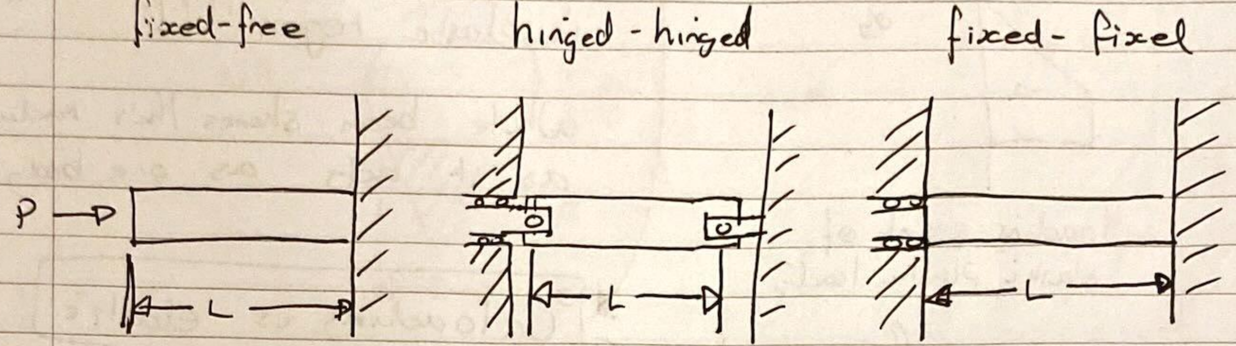
elastic instability

$\pi \approx 3$

$$P_c = \frac{\pi^2 EI}{L_{eff}}$$

critical buckling load

given in formula sheet



thermal stress and strain

$$\Delta L = \frac{FL}{AE} + \underbrace{L\alpha\Delta T} \quad ①$$

mechanical stetching or squishing

thermal expansion or contraction

$\boxed{\text{heating of a constrained bar}}$

1. set $\Delta L = 0$
2. rearrange for $F/A \ (= \sigma)$ to find stress

$\boxed{\text{finding stress distributions}}$

1. find $\bar{\varepsilon}$ from $P = \dots$ eqn in formula sheet
2. find $1/R$ from $R = \dots$ eqn in formula sheet
3. sub into $\sigma_x = \dots$ eqn
4. evaluate desired points

under no restraint or loading, $P = M = 0$

$\boxed{\text{heating of compound bar assembly}}$

&

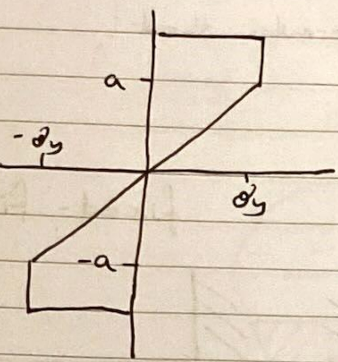
$\Delta L_1 = \Delta L_2$

$F_1 = -F_2$ ← due to equilibrium

1. equate ΔL eqns ①
2. sub $F_1 = -F_2$
3. rearrange for F_1 or F_2
4. Material with lower α is in tension

elastic plastic deformation

$\boxed{\text{1. loading + moment equilibrium}}$



loading graph of elastic plastic loading

use $M = \int_A y\sigma \, dA$

(in eqn sheet)

to find bending moment

$\boxed{\text{2. compatability (find radius)}}$

use $\frac{\sigma}{y} = \frac{E}{R}$ (in eqn sheet)

to find R.

use a value of σ and y in elastic region!!!!

Whole beam shares this radius as it acts as one body

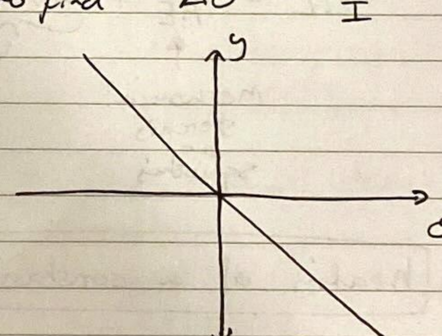
$\boxed{\text{3. unloading is elastic}}$

unloading is assumed to be elastic

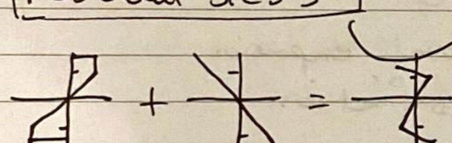
∴ you can use

$\frac{M}{I} = \frac{\sigma}{y}$ (in eqn sheet)

to find $\Delta\sigma = \frac{\Delta M y}{I}$



$\boxed{\text{residual stress 4.}}$



load + unload = residual

$\boxed{\text{residual curvature 5.}}$

the residual stress at a (where part was loaded to σ_y) ~~is in~~ was ~~elastic~~ always in elastic region so elastic bending eqn can be used

$$\frac{M}{I} = \frac{\sigma}{y}$$ (in eqn sheet)

$\boxed{\text{torsional stresses}}$

everything is the same apart from the symbols

M = T

I = J

$\sigma = \tau$

y = r

R = θ/L

E = G

Shear stresses in beams

shear force on section

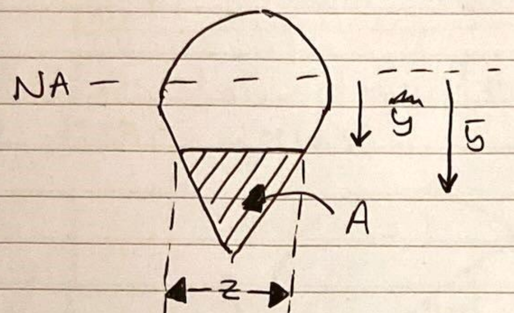
$$\tau = \frac{S}{Iz}\int y\,dA = \frac{S}{Iz}A\bar{y}$$

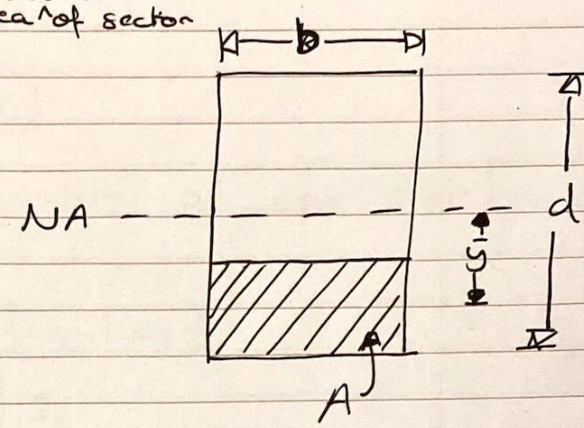
distance of section from neutral axis

second moment of area

thickness of section

outside area of section



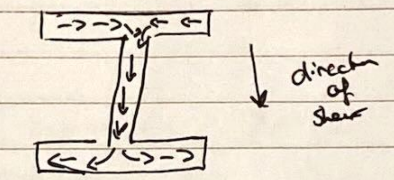


for circular beams

$$\tau = \frac{4S}{3\pi R^2}\left[1-\left(\frac{y}{R}\right)^2\right]$$

(not in eqn sheet)

dominant stress in I beam



find shear centre

1. find dominant stresses in each part
2. integrate to find shear force in each part
3. take moment of shear forces

to derive:

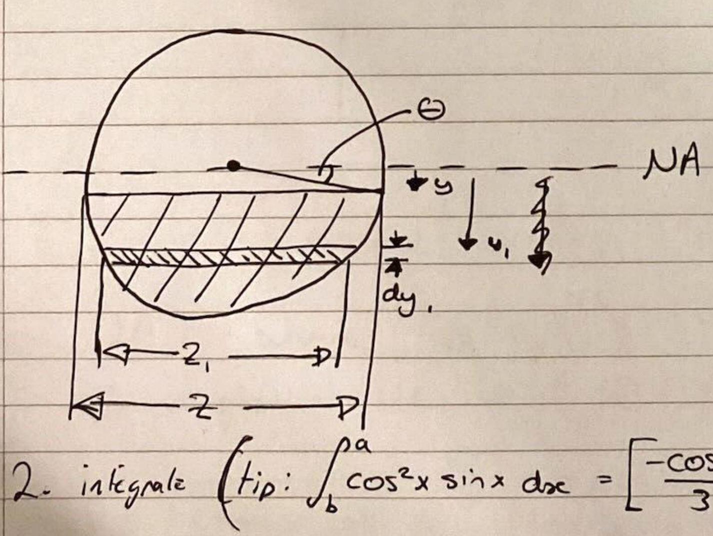
1. convert variable to polar

$y_1 = R\sin\theta_1$

$dy_1 = R\cos\theta_1 d\theta_1$

$z_1 = 2R\cos\theta_1$

$z = 2R\cos\theta$



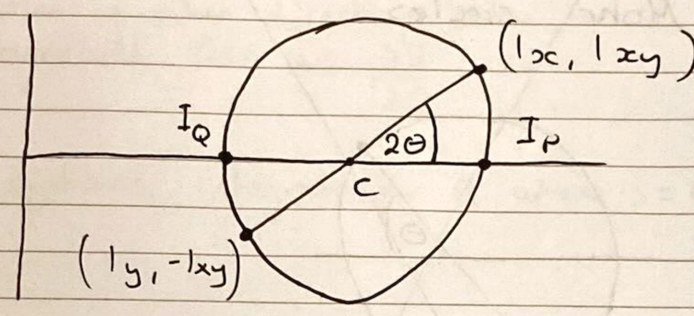
2. integrate (tip: $\int_b^a \cos^2 x \sin x\,dx = \left[\frac{-\cos^3 x}{3}\right]_b^a$)

asymmetrical bending

1. find principal axes
2. resolve bending moments onto axes
3. find neutral axis
4. find stress at furthest point from NA

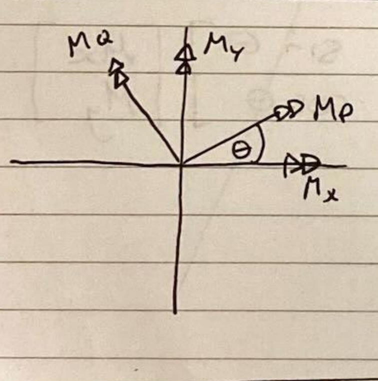
find principal axes

1. find $I_{x'x'}$, $I_{y'y'}$, $I_{x'y'}$ (equations on eqn sheet)
2. plot Mohr circles



3. I_Q, I_P are principal 2nd moments, ~~2θ~~ θ is angle of principal axes compared to x-y axes

resolve moments onto principal axes



$$M_P = M_x \cos\theta + M_y \sin\theta$$
$$M_Q = -M_x \sin\theta + M_y \cos\theta$$

or

$$\begin{bmatrix} M_P \\ M_Q \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} M_x \\ M_y \end{bmatrix}$$

find neutral axis

NA is where $\sigma = \frac{M_P}{I_P} Q - \frac{M_Q}{I_Q} P = 0$

$\therefore$ angle of NA, $\alpha = \tan^{-1}\frac{Q}{P} = \tan^{-1}\frac{M_Q I_P}{M_P I_Q}$

stress at a given point

$$\sigma = \frac{M_P}{I_P} Q - \frac{M_Q}{I_Q} P$$

Strain Energy

Axial:

$$U = \int_0^L \frac{P^2}{2AE}\,dx$$

Bending:

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

Torsion:

$$U = \int_0^L \frac{T^2}{2GJ}\,dx$$

$$u = \frac{\partial U}{\partial P}$$

Method is to divide case into sections and integrate to obtain strain energy for whole beam. $U = U_{AB} + U_{BC}$ etc.

Then use $u = \frac{\partial U}{\partial P}$ to find deflections at location of loading (and direction).

To find deflection in other direction, solve for U for a "dummy" load with no magnitude, then use $\frac{\partial U}{\partial Q} = u_q$

For Circular systems, integrate for θ, where $s = R\theta$.

Thick Walled Cylinders

Recall from thin walled cylinders: $\sigma_\theta = \frac{PR_m}{t}$ $\quad \sigma_z = \frac{PR_m}{2t}$

For thick walled we use Lame's equations:

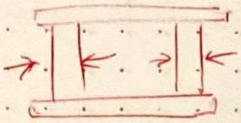
All in Equation Sheet

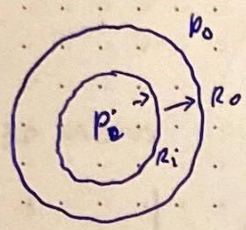
$$\sigma_\theta = A + \frac{B}{r^2} \qquad \sigma_r = A - \frac{B}{r^2}$$

r can be used to find stress at any radius

$$\sigma_z = \frac{R_i^2 p_i - R_o^2 p_o}{R_o^2 - R_i^2}$$

$\sigma_z = 0$ if pistons are present.





By knowing p_i and p_o you can solve for σ_r at R_i and R_{out} as $\sigma_r = -p$. ∴ Finding A and B with simult. equations.

For most questions this is enough.

May need to solve for multiple compatible cylinders

Strain ⇒ $\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right)$

$\varepsilon_z = \frac{\Delta L}{L} = \frac{1}{E}\left(\sigma_z - \nu(\sigma_\theta + \sigma_r)\right)$

} Generalised Hookes Law

For compatability, may have to equate strains or sum of strains/deflections.

Rotating Discs:

$$\sigma_r = A - \frac{B}{r^2} - \frac{\rho\omega^2(3+\nu)}{8} r^2$$

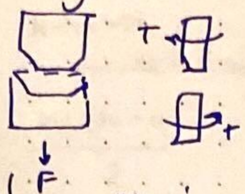
$$\sigma_\theta = A + \frac{B}{r^2} - \frac{\rho\omega^2(1+3\nu)}{8} r^2$$

Same method as for pressure vessels.

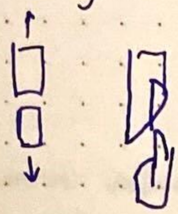
Yield Criteria

Yielding cause % elongation = $\frac{l_f - l_0}{l_0}$ and % area reduction $\frac{A_0 - A_f}{A_0}$

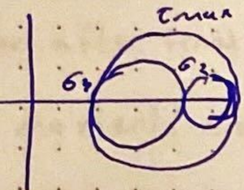
Ductile failure => along maximum shear stress plane



Brittle failure => along maximum principle stress plane



3D Mohrs



Tresca Criterion:

$$\sigma_1 - \sigma_3 \geq \sigma_y^2 \quad \text{for } \sigma_1 > \sigma_2 > \sigma_3$$

Von Mises:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2 \sigma_y^2$$

Yield criteria are independent of hydrostatic stress $\sigma_h = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

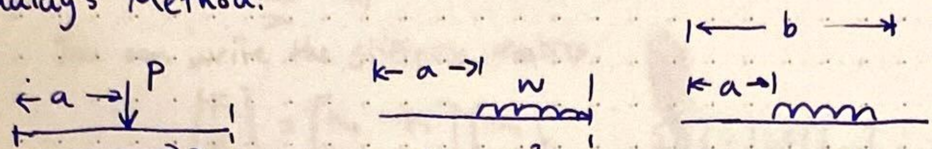
To answer questions:

- Draw 2D stress state, draw Mohr's circle
- Determine τ_{max}, σ_1, σ_2, σ_3
- Use either Tresca or Von Mises to solve

Deflection of Beams

$$M = EI \frac{d^2y}{dx^2} \quad ①$$

Macaulay's Method:



$P\langle x-a\rangle$

Point Load

$\dfrac{w\langle x-a\rangle^2}{2}$

Continuous UDL

Discontinuous UDL

$\dfrac{w\langle x-b\rangle^2}{2} - \dfrac{w\langle x-a\rangle^2}{2}$

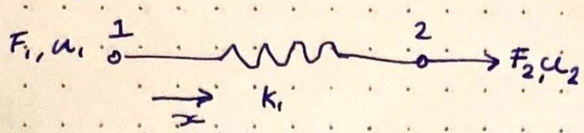
Point Moment: $M_0\langle x-a\rangle^0$

Method:

- FBD to determine values of R_A, R_B if possible
- Section after final discontinuity and draw FBD
- Take moments around section position M s s M
- Obtain $M(x)$ and sub into ①
- Integrate for $EI\frac{dy}{dx}$ and EIy DONT FORGET $\int$ constants
- Use boundary conditions to solve for A and B

FEA

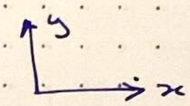
For a single 1D element:

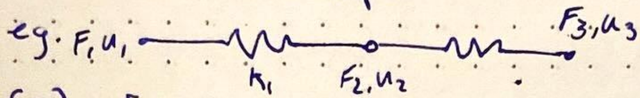


You can write the stiffness matrix:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \{F\} = [k^e]\{u\}$$

You can then combine multipe:





$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

For a 2D truss element:

$$[k^e]_g = \left(\frac{AE}{L}\right) \begin{bmatrix} c^2 & cs & -c & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Same method as 1D trusses:
- Make stiffness matrices
- Sum into 1 big one
- Eliminate a bunch from BC's

Note. θ is anticlockwise from x axis along direction of truss.

Approximations:

Plane-stress:

A thin plate only loaded in-plane. Can assume $\sigma_z = 0$

Plane-Strain:

Very thick/long plate only loaded in plane. $\varepsilon_z = 0$ $\therefore$ $\sigma_z = \nu(\sigma_x + \sigma_y)$

Axisymmetric

Rotationally symmetric cases. σ's are independent of θ

Symmetry