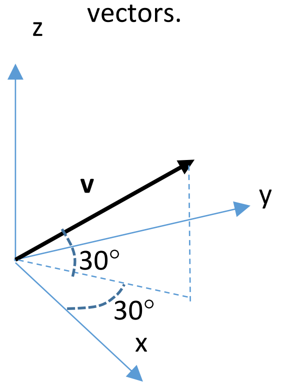
# Questions for Navier Stokes derivation section

1. Express the 3D velocity vector, $\boldsymbol{v}$, with magnitude 30 m/s, in terms of scalar speed terms and the direction vectors.




2. Explain how it is that in a 3D cuboid considering the Eulerian frame of reference, with incompressible liquid flow, the velocity going in one face in the x-direction may not be the same as the velocity going out of the opposite face.
3. Expand the continuity equation $div\ \boldsymbol{u} = 0$ into separate terms for the three coordinate directions.
4. For the 3 stress components on a face of a cuboid with moving incompressible fluid passing through it, with the aid of a sketch explain the meaning of the subscript terms on $\sigma_{zz}$, $\tau_{zx}$ and $\tau_{zy}$.
5. Briefly state how solid mechanics direct stress is similar and different to fluid mechanics direct stress and name the relevant laws of physics pertaining to each one. What aspect of flow causes the direct stress?
6. Gather the terms from:

$$F_z = \left( -\frac{\partial p}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial z^2}\right) + 2\mu\frac{\partial^2 w}{\partial z^2} + \mu\left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \mu\left(\frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right)\delta x \delta y \delta z$$

   To derive:

$$F_z = \left( -\frac{\partial p}{\partial z} + \frac{1}{3}\mu\left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial z^2}\right) + \mu\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right)\delta x \delta y \delta z$$

7. Use vector identities to prove that $\mu\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$ is equivalent to $\mu\nabla^2\mathbf{u}$.
8. Use vector identities to prove that $\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z}$ is equivalent to $(\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\otimes\boldsymbol{u})^T$.
9. Use the equation of continuity to demonstrate that $\frac{1}{3}\mu\ grad\ div\ \boldsymbol{u}$ is zero for incompressible flow.
10. Name the three terms of the incompressible Navier-Stokes equation and briefly describe their meaning.

11. What is the shear stress adjacent to a smooth wall with air flowing over it with a laminar velocity gradient at the wall of 4454 $s^{-1}$? Viscosity of water can be taken as $1.846\times10^{-5}$ $kgm^{-1}s^{-1}$.
12. What is the pressure at the end of a 12mm diameter hose, 5m below the free surface of a header tank supplying water at 300 mL/s to the hose and with a frictional coefficient on the pipe walls of 0.001 in a 10m length of pipe? (Use $\Delta p = \frac{4fL\rho v^2}{2d}$ for pipe friction loss, $\Delta p = \frac{1}{2}\rho v^2$ for the dynamic pressure loss, and $v = \frac{\dot{V}}{A}$ for the pipe velocity, with g=9.81 m/s$^2$).
13. Starting with the tensor expression of the Navier Stokes incompressible form of the equation for fluid flow, expand to the Cartesian matrix form of the Navier-Stokes equations in 2D (x and y directions only).
14. From your understanding of differential equations, what is it about Navier-Stokes that is difficult.

# Solutions to Questions on Navier-Stokes

1. x component: v.cos30.cos30 = 0.75v

   ~~y~~ component: v.sin30 = 0.5v  this is z component

   ~~z~~ component: v.cos30.sin30 = 0.433v  this is y component

   Error on this solution in type: the z and y components were swapped over by mistake Corrected in blue text.

   0.43  0.5

   $$\vec{v} = 0.75v\boldsymbol{i} + \cancel{0.50}v\boldsymbol{j} + \cancel{0.43}v\boldsymbol{k}$$

   $$\vec{v} = 22.5\boldsymbol{i} + \cancel{15}v\boldsymbol{j} + \cancel{12.9}v\boldsymbol{k}$$

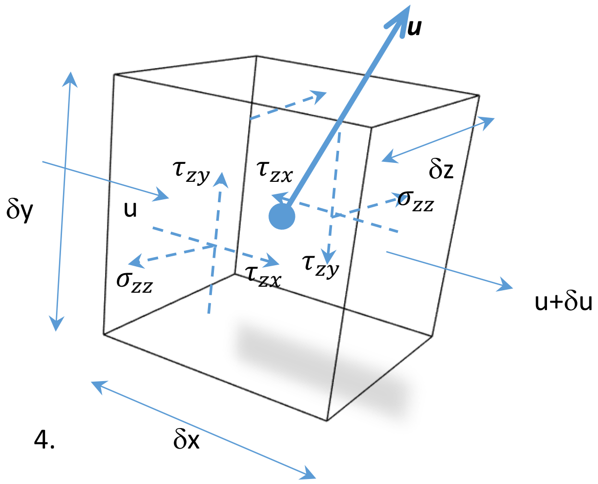
   12.9  15

2. In 3D flow, there will be flow into the cube from all 3 axis directions. Therefore it is possible that there will be diversion of the flow from the x-direction into the other axis directions, or from the other axis directions into the x-direction. And in this way x velocity may not be the same at one side as at the other.

3. $div\ \boldsymbol{u} = 0$ is expressed in full as:

$$\frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial v}{\partial y}\boldsymbol{j} + \frac{\partial w}{\partial z}\boldsymbol{k} = 0$$

4.




As can be seen from the figure, the stresses with first subscript z act on the face which is perpendicular to the z direction. The second subscript is the direction in which the stress acts on that face.

5. In solid mechanics, direct stress is caused by a tensile (pulling or pushing) force on the material, and since the material is structurally connected, with no significant relative motion between atoms, then the stress is directly related to the force. Although there is some stretching of the atoms related to the Young's modulus or to plastic deformation, there is no need for motion to cause the stress response. In fluid mechanics, direct stress is due to stretching of the fluid, which can only happen due to viscous action when regions of fluid slide over each other, and so although the stress is in the same sense as solid mechanics, it is due to motion of atoms within the fluid.

6.

$$F_z = \left(-\frac{\partial p}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial z^2}\right) + 2\mu\frac{\partial^2 w}{\partial z^2} + \mu\left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \mu\left(\frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right)\delta x \delta y \delta z$$

Extract the pressure and the last term:

$$F_z = \left( -\frac{\partial p}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial z^2}\right) + \mu\frac{\partial^2 w}{\partial z^2} + \mu\frac{\partial^2 u}{\partial z\partial x} + \mu\frac{\partial^2 v}{\partial z\partial y} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\right)\delta x\delta y\delta z$$

Note that we needed to take one of the two $2\mu\frac{\partial^2 w}{\partial z^2}$ term to satisfy the last bracket.

And now it is clear to see that when the other terms are gathered, you get the middle term of the expression required.

$$F_z = \left( -\frac{\partial p}{\partial z} + \left(1 - \frac{2}{3}\right)\mu\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial z^2}\right) + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\right)\delta x\delta y\delta z$$

$$F_z = \left( -\frac{\partial p}{\partial z} + \frac{1}{3}\mu\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial z^2}\right) + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\right)\delta x\delta y\delta z$$

7. $\mu\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$

We know that $\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$

And we know that $\nabla^2 = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot\left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)$

which is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If we now operate on velocity vector ***v*** with this operator:

$$\nabla^2\boldsymbol{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\cdot(\boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w)$$

We get

$$\nabla^2\boldsymbol{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\cdot(\boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w)$$

$$\nabla^2\boldsymbol{u} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\boldsymbol{i} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\boldsymbol{j} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\boldsymbol{k}$$

and the last term is the k-direction term which is the required point of proof.

8. $\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z}$

We know that:

$$\rho\big((\boldsymbol{u}\cdot\boldsymbol{\nabla})\otimes\boldsymbol{u}\big)^T = \left([u\boldsymbol{i} \quad v\boldsymbol{j} \quad w\boldsymbol{k}]\cdot\left[\frac{\partial}{\partial x}\boldsymbol{i} \quad \frac{\partial}{\partial y}\boldsymbol{j} \quad \frac{\partial}{\partial z}\boldsymbol{k}\right]\otimes[u\boldsymbol{i} \quad v\boldsymbol{j} \quad w\boldsymbol{k}]\right)^T$$

which expands to

$$\rho\left((\boldsymbol{u}\cdot\boldsymbol{\nabla})\otimes\boldsymbol{u}\right)^{\boldsymbol{T}} = \left(\begin{bmatrix} u\frac{\partial}{\partial x} & v\frac{\partial}{\partial y} & w\frac{\partial}{\partial z}\end{bmatrix}\otimes[u\boldsymbol{i} \quad v\boldsymbol{j} \quad w\boldsymbol{k}]\right)^{T}$$

And applying the outer product:

$$\begin{bmatrix} u\frac{\partial}{\partial x} \\ v\frac{\partial}{\partial y} \\ w\frac{\partial}{\partial z}\end{bmatrix}\otimes[u \quad v \quad w] = \begin{bmatrix} u\frac{\partial u}{\partial x} & u\frac{\partial v}{\partial x} & u\frac{\partial w}{\partial x} \\ v\frac{\partial u}{\partial y} & v\frac{\partial v}{\partial y} & v\frac{\partial w}{\partial y} \\ w\frac{\partial u}{\partial z} & w\frac{\partial v}{\partial z} & w\frac{\partial w}{\partial z}\end{bmatrix}$$

And finally transpose:

$$\begin{bmatrix} u\frac{\partial u}{\partial x} & v\frac{\partial u}{\partial y} & w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} & v\frac{\partial v}{\partial y} & w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} & v\frac{\partial w}{\partial y} & w\frac{\partial w}{\partial z}\end{bmatrix}$$

The top line of which is the x-direction component required as the point of proof.

9. $\frac{1}{3}\mu\ grad\ div\ \boldsymbol{u}$ and we know that continuity for incompressible flow means that *div**u*** is zero. Therefore for incompressible flow, this term is zero.

10. The three terms in the incompressible fluid Navier-Stokes equations are: convection, indicating the carrying of a fluid property (temperature, momentum, turbulence) along with the velocity of the fluid in the axis directions; pressure, indicating the driving influence of the static pressure variation in the fluid flow; diffusion, indicating the dilution of a fluid property (temperature, momentum, turbulence) by diffusive effects (thermal diffusivity, viscous action, and turbulent viscosity).

11. What is the shear stress adjacent to a smooth wall with air flowing over it with a laminar velocity gradient at the wall of 4454 $s^{-1}$? Viscosity of water can be taken as $1.846\times10^{-5}$ $kgm^{-1}s^{-1}$.

Using Newton's law of viscosity:

$$\tau = \mu\frac{\partial u}{\partial y} = 1.846\times10^{-5}\times 4454 = 0.082\ N/m^2$$

12. What is the pressure at the end of a 12mm diameter hose, 5m below the free surface of a header tank supplying water at 300 mL/s to the hose and with a frictional coefficient on the

pipe walls of 0.001 in a 10m length of pipe? (Use $\Delta p = \frac{4fL\rho v^2}{2d}$ for pipe friction loss, $\Delta p = \frac{1}{2}\rho v^2$ for the dynamic pressure loss, and $v = \frac{\dot{V}}{A}$ for the pipe velocity, with g=9.81 m/s$^2$).

Carry out an audit of the pressure effects, according to a Bernoulli type analysis:

Hydrostatic term: $\rho gz = 1000 \times 9.81 \times 5 = 49050\ Pa$

Dynamic head: $\frac{1}{2}\rho v^2 = \frac{1}{2} \times 1000 \times u^2$

Need the velocity, take a 1D profile (i.e. assume that the velocity in the pipe is not influenced by a boundary layer and the velocity is the same across the entire diameter):

Velocity: $v = \frac{vol\ rate}{cross\ section} = \frac{300 \times 10^{-6}}{0.006^2 \pi} = 2.7\ m/s$

And continuing dynamic head is: $\frac{1}{2} \times 1000 \times u^2 = 3645\ Pa$

Pipe friction using: $\frac{4fL\rho v^2}{2d} = \frac{4 \times 0.001 \times 10 \times 1000 \times 2.7^2}{2 \times 0.012} = 12155\ Pa$

Therefore the net pressure at the end of the pipe will be the hydrostatic head, minus the dynamic head (due to turning pressure into velocity), and minus the pipe friction loss: 49050-3645-12155=33250 Pa

13. Starting with the tensor expression of the Navier Stokes incompressible form of the equation for fluid flow, expand to the Cartesian matrix form of the Navier-Stokes equations in 2D (x and y directions only).

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

Cycling the free index, *i*, in x and y, and for each one, cycling the dummy index, *j*, produces:

Free index i=x direction, and u velocity

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

Free index i=y direction and v velocity

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}$$

Which are the 2D, incompressible, Navier-Stokes equations.

14. From your understanding of differential equations, what is it about Navier-Stokes that is difficult.

2nd order (first difficulty but solvable), 3D (second difficulty but solvable), inhomogeneous (3rd difficulty, mostly unsolvable by analytical means – numerical methods required).