University of Nottingham
Department of Mechanical, Materials and Manufacturing Engineering

Computer Modelling Techniques

FE-01-02

STRESS ANALYSIS FUNDAMENTALS

Lecture Outline

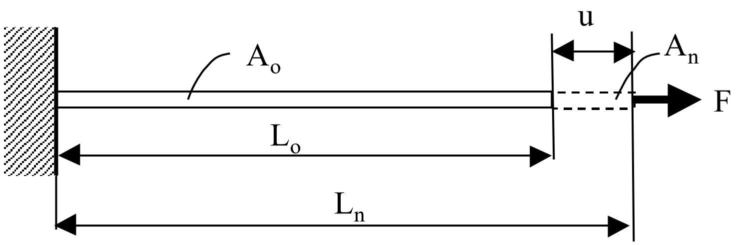
1.1 Uniaxial Loading

1.1.1 Uniaxial (one-dimensional) Stress and Strain Relationships

For a uniaxial loading situation, conventional (engineering) strain is defined as the change in length per unit original (undeformed) length.

For a long bar under a uniaxial load, the engineering (or nominal) strain is:

$$\varepsilon_{engineering} = \frac{L_n - L_o}{L_o} = \frac{u}{L_o}$$



For a uniaxial loading situation, the stress can be simply defined as follows:

$$\sigma_{engineering} = \frac{F}{A_o}$$

This definition assumes that the stress is uniform over that particular area, but in reality stresses are seldom uniform over large areas.

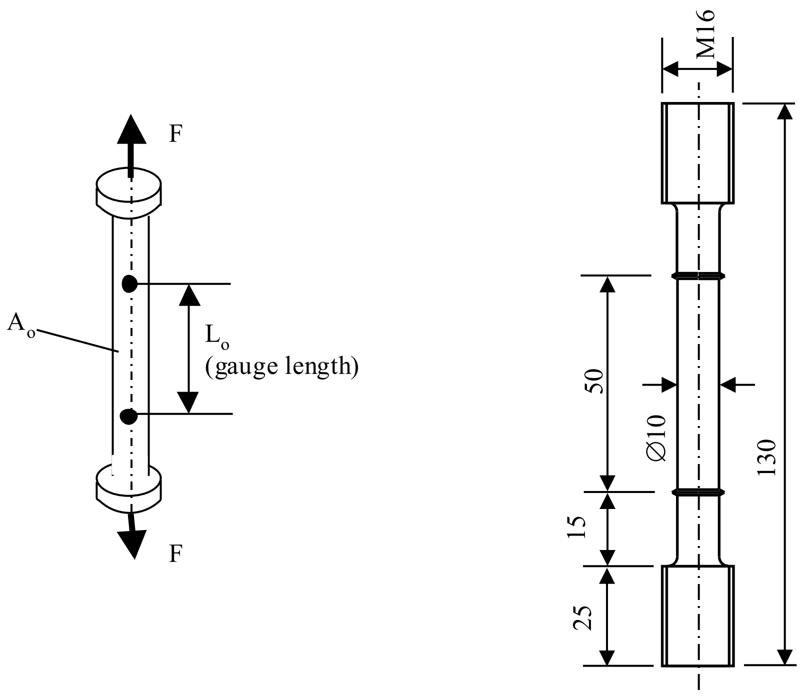
Therefore, it is more meaningful if this area is made very small, thus introducing the mathematical concept of *"stress at a point"* defined as follows:

$$\sigma = Lim_{\delta A \to 0} \left(\frac{\delta F}{\delta A} \right)$$

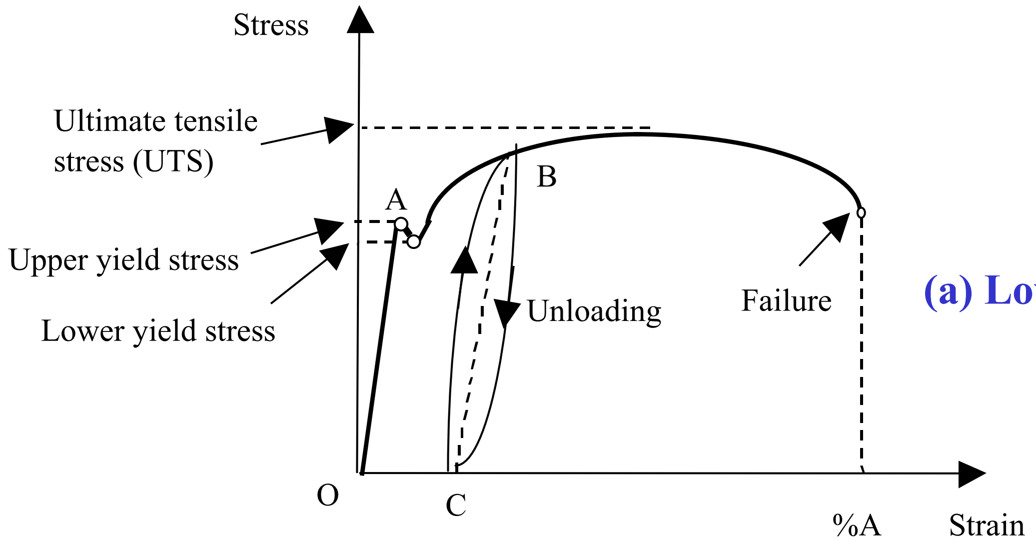
The concept of stress at a point is physically valid because a small area *dA* would carry a small amount of force *dF*.

1.1.2 Uniaxial stress-strain curves

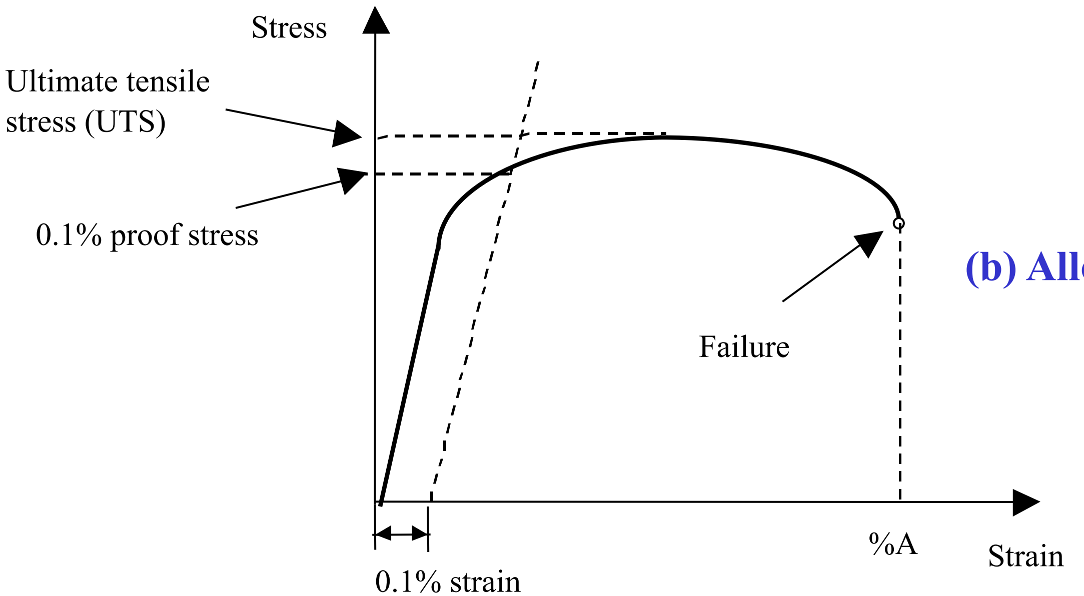
Some basic ideas underlying the theory of elasticity and plasticity can be presented with reference to simple one-dimensional experimental tests on an elasto-plastic material, in which a test specimen in the shape of a cylindrical bar is subjected to a uniaxial tension *F*.



Uniaxial test specimen



(a) Low and medium carbon steels



(b) Alloy steels and non-ferrous metals

Typical uniaxial stress-strain curves

1.2 Three-Dimensional Stress and Strain

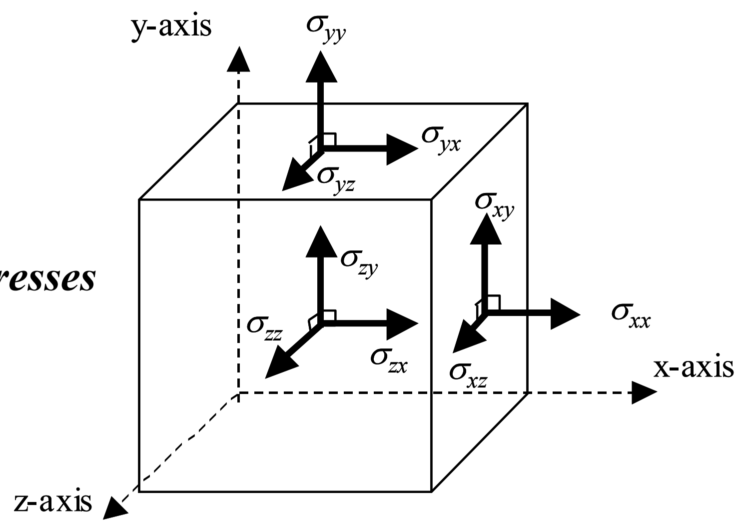
1.2.1 Multi-axial (3D) stress definitions

In a 3D Cartesian axes system, there are six components of stress, as follows:

- Three **direct (tensile or compressive)** stresses (σ_{xx} , σ_{yy} , σ_{zz}) caused by forces normal to the area

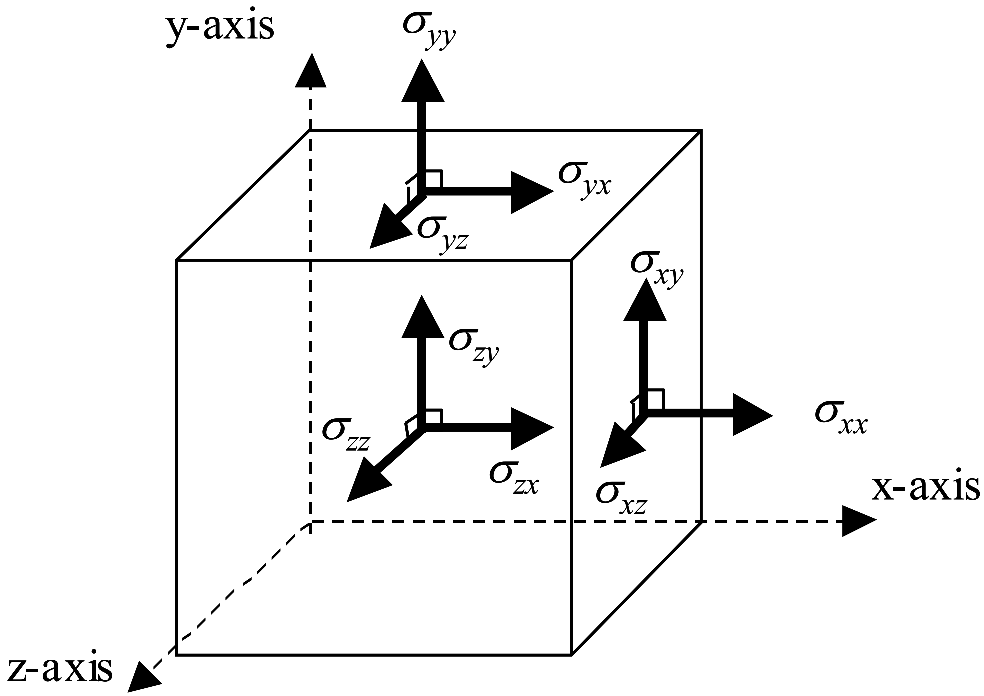
- Three **shear stresses** (σ_{xy} , σ_{xz} , σ_{yz}) caused by shear forces acting parallel to the area

3D Cartesian stresses



The first subscript refers to the direction of the outward normal to the plane on which the stress acts, and the second subscript refers to the direction of the stress arrow

For simplicity, in most problems the first and second subscripts can be interchanged, i.e. $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{xz} = \sigma_{zx}$ (complementary shear stress).



A **“stress matrix”** or a **“stress vector”**, which contains all stress components, can be conveniently expressed as follows:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

Similarly, a **“strain vector”** can be defined as follows:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix}$$

1.2.2 3D strain definitions

- For 3D multi-axial problems, the uniaxial definition of strain as $\Delta L/L$ is not applicable. 3D strains can only be defined in terms of the displacements of the domain.

- The **3D direct (i.e. non-shear) strains** are related to the displacement components as follows:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

- Note that the use of partial differentiation symbol (∂) in the above equations is intentional and indicates that the displacements (u_x, u_y and u_z) can be functions of the x, y and z coordinates.

- The notation and sign convention used for strains are the same as those used for stresses, i.e. tensile strain (elongation) is positive while compressive strain is negative.

The **3D shear strains** are defined as follows:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

The ½ factor in shear strain

- Two definitions are often used for the shear strain, one with the 1/2 factor and one without.
- The shear strain definition with the 1/2 factor (often referred to as the *"**mathematical shear strain**"*) is mainly used for the convenience of use in tensor notations.
- The shear strain definition without the 1/2 factor is referred to as the *"engineering shear strain"*.
- Both definitions are valid, provided that the definition is followed throughout the derivation of other relationships involving strains.

1.2.3 3D Stress-Strain Relationships (Hooke's Law)

Stress-strain relationships are often called *"Constitutive Equations"*. For isotropic linear elastic materials with thermal strain, the following 3D stress-strain equations (Hooke's law) can be used:

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] + \alpha\left(\Delta T\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] + \alpha\left(\Delta T\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \alpha\left(\Delta T\right)$$

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

$$\varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz}$$

$$\varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}$$

where
E = Young's modulus (units: N m^{-2})
ν = Poisson's ratio (no units)
μ = Shear modulus (units: N m^{-2})
α = Coefficient of thermal expansion (units: per ^{o}C)
ΔT = Temperature change from a reference value (units: ^{o}C)

The shear modulus μ is defined as follows:

$$\mu = \frac{E}{2\left(1 + \nu\right)}$$

In computational mechanics formulations, it is often more convenient to place stresses on the left hand side of the equations, i.e. stresses expressed as functions of strains.

$$[\sigma] = [D]\,[\varepsilon]$$

where $[D]$ is called the *"elastic property matrix"*.