University of Nottingham
Department of Mechanical, Materials and Manufacturing Engineering

Computer Modelling Techniques

FE-01-04

MATHEMATICAL BACKGROUND

1.4 Some Mathematical Background on Matrices

Using Matrices to represent equations

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1$$
$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2$$
$$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3$$

can be expressed as matrices as follows:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or, in a more concise form:

$$[A]\,[x] = [b]$$

where, in this example, $[A]$ is a 3 x 3 matrix, $[x]$ and $[b]$ are 3 x 1 matrices.

If N is the total number of equations, then $[A]$ is a N x N matrix, $[x]$ and $[b]$ are N x 1 matrices. Note that matrices such as $[x]$ and $[b]$ with just one column are sometimes called ***"vectors"***.

Matrix Multiplication

In general, if two matrices [*A*] and [*B*] are multiplied, then the number of columns of [*A*] must be the same as the number of rows of [*B*], i.e. if [*A*] is a (m x n) matrix, and [*B*] is a (p x q) matrix, then n must be equal to p.

The resulting matrix [*C*] is a (m x q) matrix.

$$[A]^{(m \times n)} \times [B]^{(p \times q)} = [C]^{(m \times q)} \quad \ldots. (n \text{ must be equal to } p)$$

Note that, in general, [*A*] x [*B*] is not equal to [*B*] x [*A*]

Important

Transpose of a Matrix

If the rows and columns of a matrix $[A]$ are interchanged

The following example shows a matrix $[A]$ and its transpose $[A]^T$:

$$[A] = \begin{bmatrix} 4 & 2 & 6 & 9 \\ 3 & 7 & 8 & 2 \\ 17 & 5 & 5 & 11 \\ 22 & 7 & 8 & 1 \end{bmatrix} ; \; [A]^T = \begin{bmatrix} 4 & 3 & 17 & 22 \\ 2 & 7 & 5 & 7 \\ 6 & 8 & 5 & 8 \\ 9 & 2 & 11 & 1 \end{bmatrix}$$

The following relationships are useful:

$$([A] \times [B])^T = [B]^T \times [A]^T$$

$$\left([A]^T\right)^T = [A]$$

Symmetric Matrix

A square matrix (number of rows equal to the number of columns) is called ***"symmetric"*** if $[A]^T = [A]$, i.e. $a_{ij} = a_{ji}$.

This means that matrix coefficients above the diagonal of the matrix are "mirror images" of those below the diagonal.

For example, the following square (4x4) matrix is symmetric:

$$[A] = \begin{bmatrix} 4 & 2 & 6 & 9 \\ 2 & 7 & 8 & 2 \\ 6 & 8 & 5 & 7 \\ 9 & 2 & 7 & 1 \end{bmatrix}$$

In FE formulations, the stiffness matrices are symmetrical, and it is important to exploit this symmetry to economise on the storage requirements of large matrices.

Inverse of a Matrix

A ***"unit matrix"***, [I], is a square matrix in which all the coefficients of the principal diagonal are equal to 1, while all other coefficients are zero, as follows:

$$[I]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

If for a given matrix [A] there exists a matrix [B] such that [A][B] = [I], where [I] is a *"unit matrix"*, then [B] is called the "inverse" of [A] and is denoted by $[A]^{-1}$ as follows:

$$[A]\times\left[A^{-1}\right]=[I]$$

Therefore, to solve a system of linear algebraic equations, both sides of the equation can be multiplied by $[A]^{-1}$ to give:

$$[x]=\left[A^{-1}\right][b]$$

- Inverting a large matrix requires a substantial number of mathematical operations, e.g. of the order of N^4 where N is the number of equations.

Important

- In practice, the direct computation of the inverse of $[A]$ is avoided, because it is very "expensive" (i.e. requires a substantial amount of computational time).

- Instead, special equation solving methods such as **"*Gaussian Elimination*"** or iterative **"*Gauss-Seidel*"** techniques are used. *(This will be covered later in the module)*

An Example of Using Matrices in Equations

Consider a one-dimensional (uniaxial) problem where the strain energy stored in the body, per unit volume, is given by:

$$U = \frac{1}{2} \sigma_{xx} \, \varepsilon_{xx}$$

This expression can be generalised for two-dimensional problems, as follows:

$$U = \frac{1}{2} \left(\sigma_{xx} \, \varepsilon_{xx} + \sigma_{yy} \, \varepsilon_{yy} + \sigma_{xy} \, \varepsilon_{xy} \right)$$

Similarly, for three-dimensional problems:

$$U = \frac{1}{2} \left(\sigma_{xx} \, \varepsilon_{xx} + \sigma_{yy} \, \varepsilon_{yy} + \sigma_{zz} \, \varepsilon_{zz} + \sigma_{xy} \, \varepsilon_{xy} + \sigma_{xz} \, \varepsilon_{xz} + \sigma_{yz} \, \varepsilon_{yz} \right)$$

Using matrices, all 3 equations can be combined as follows:

$$U = \frac{1}{2} [\sigma]^T [\varepsilon]$$

where $[\sigma]$ and $[e]$ are the stress and strain vectors.

An Alternative Tensor Notation

An alternative notation, called the ***"tensor"*** notation, is also widely used in computational mechanics formulations. This notation is based on using subscripts such as *i*, *j*, and *k* as follows:

$$U = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

where the subscripts *i* and *j* take the values 1, 2 and 3 corresponding to the Cartesian directions *x*, *y* and *z*, respectively.

Both matrix and tensor notations are widely used in computational mechanics formulations.

Only matrix expressions will be used hereafter.