MTHS2007 Advanced Mathematics and Statistics for Mechanical Engineers

Chapter 5: Laplace transforms

School of Mathematical Sciences

The University of Nottingham

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5 Laplace Transforms

This chapter is about a very powerful technique for differential equations (and some integral equations): **Laplace transforms**. It is the language of many areas of engineering, such as control theory.

ODE for $y(t)$	(i) transform $\longrightarrow$ $\bar{y} = \mathcal{L}(y)$	algebraic equation for $\bar{y}(s)$
		$\downarrow$ (ii) solve for $\bar{y}(s)$
solution $y(t)$	(iii) inverse transform $\longleftarrow$ $y = \mathcal{L}^{-1}(\bar{y})$	$\bar{y}(s) =$ known

The approach is not to solve the ODE directly, but to transform the solution $y(t)$ into a new function of a new variable $\bar{y}(s)$ which solves an (easier!) algebraic problem.

5.1 Definition and basic properties

Given a function $f(t)$, defined for $t \geq 0$, its *Laplace transform* is written

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

provided this integral exists. If necessary we will assume that s is (positive and) large enough for this to be the case.

Example Let

$$f(t) = e^{at}.$$

Then

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \bar{f}(s) &= \int_0^\infty e^{-st} e^{at}\,dt \\
&= \int_0^\infty e^{-(s-a)t}\,dt \\
&= \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}
\end{aligned}$$

provided $a < s$.

To see why this is useful, let us see how it transforms derivatives.

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} &= \int_0^\infty e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t \\ &= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty (-se^{-st})f(t)\,\mathrm{d}t \\ &= -f(0) + s\int_0^\infty e^{-st}f(t)\,\mathrm{d}t \\ &= -f(0) + s\bar{f}(s)\end{aligned}$$

Notice:

- We must assume that s is large enough and $f(t)$ grows slowly enough that $e^{-st}f(t) \to 0$ as $t \to \infty$ for this to be true.
- A derivative (calculus) has been turned into multiplication (algebra).
- Furthermore, initial conditions are automatically accounted for!

Similarly, we can obtain the Laplace transform of the second derivative. Use the previous result

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = -f(0) + s\bar{f}(s) = -f(0) + s\mathcal{L}\{f(t)\}$$

with $f(t) \to f'(t)$ to get

$$\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right\} &= \mathcal{L}\left\{\frac{\mathrm{d}f'}{\mathrm{d}t}\right\} \\
&= -f'(0) + s\mathcal{L}\{f'(t)\} \\
&= -f'(0) + s(-f(0) + s\mathcal{L}\{f(t)\} \\
&= -f'(0) - sf(0) + s^2\mathcal{L}\{f(t)\} \\
&= -f'(0) - sf(0) + s^2\bar{f}(s)
\end{aligned}$$

We can keep going to higher derivatives! Eg.

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}^3 f}{\mathrm{d}t^3}\right\} &= \mathcal{L}\left\{\frac{\mathrm{d}f''}{\mathrm{d}t}\right\} \\ &= -f''(0) + s\mathcal{L}\{f''(t)\} \\ &= -f''(0) + s(-f'(0) - sf(0) + s^2\bar{f}(s)) \\ &= -f''(0) - sf'(0) - s^2 f(0) + s^3\bar{f}(s).\end{aligned}$$

These results are useful for solving ODEs, as the Laplace transforms include no derivatives of $\bar{f}(s)$: If we apply the Laplace transform to an ODE, we will get a purely algebraic equation.

Advantages:

(a) Initial conditions are built in from the start. This means the method is particularly suitable for initial value problems, where f and f' are known at $t = 0$.

(b) There is no need to guess a particular integral.

Example

Let $f(t) = 1$. Then

$$\bar{f}(s) = \int_0^\infty e^{-st}\, dt = \left[-e^{-st}/s\right]_0^\infty = \frac{1}{s}$$

provided $s > 0$ (otherwise the Laplace transform would not exist). Hence

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \text{and} \qquad 1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}.$$

Remark

This is a special case of the identity

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{and} \qquad e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

already shown.

Example

Let $f(t) = t$. Then

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty t e^{-st}\,dt \\
&= \int_0^\infty t \frac{d}{dt}\left(-\frac{1}{s}e^{-st}\right)\,dt \\
&= \left[t\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\
&= 0 + \frac{1}{s}\mathcal{L}(1) \qquad \text{(choose } s > 0 \text{ so brackets vanish)} \\
&= \frac{1}{s} \times \frac{1}{s} \\
&= \frac{1}{s^2}.
\end{aligned}$$

Therefore

$$\mathcal{L}\{t\} = \frac{1}{s^2}.$$

Example

Let $f(t) = t^2$. Then

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty t^2 e^{-st}\,dt \\
&= \int_0^\infty t^2 \frac{d}{dt}\left(-\frac{1}{s}e^{-st}\right)\,dt \\
&= \left[t^2\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty (2t)e^{-st}\,dt \\
&= 0 + \frac{2}{s}\mathcal{L}(t) \qquad \text{(choose } s > 0 \text{ as before)} \\
&= \frac{2}{s} \times \frac{1}{s^2} \\
&= \frac{2}{s^3}.
\end{aligned}$$

Therefore

$$\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}.$$

More generally

$$\mathcal{L}\left\{t^n\right\} = \frac{n(n-1)\cdots 2\cdot 1}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

Example

Let $f(t) = \sin(kt)$. Then

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty e^{-st}\sin(kt)\,dt = \int_0^\infty \sin(kt)\frac{d}{dt}\left(-\frac{1}{s}e^{-st}\right)\,dt \\
&= \left[\sin(kt)\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty k\cos(kt)e^{-st}\,dt \\
&= 0 + \frac{k}{s}\int_0^\infty \cos(kt)e^{-st}\,dt \qquad \text{(choose } s > 0\text{)} \\
&= \frac{k}{s}\left[\cos(kt)\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty - \frac{k^2}{s^2}\int_0^\infty \sin(kt)e^{-st}\,dt \\
&= \frac{k}{s^2} - \frac{k^2}{s^2}\bar{f}(s).
\end{aligned}$$

Hence $\left(1 + \frac{k^2}{s^2}\right)\bar{f}(s) = \frac{k}{s^2}$, which gives $\bar{f}(s) = \frac{k}{s^2 + k^2}$.

Example

Let $f(t) = \cos(kt)$. Then (alternative method) use

$$\cos(kt) = \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}kt}\right) = \mathrm{Re}\left(\cos(kt) + \mathrm{i}\sin(kt)\right)$$

to note that

$$\begin{aligned}\bar{f}(s) &= \mathrm{Re}\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t \\ &= \mathrm{Re}\int_0^\infty \mathrm{e}^{-(s-\mathrm{i}k)t}\,\mathrm{d}t \\ &= \mathrm{Re}\left(\frac{1}{s-\mathrm{i}k}\right)\end{aligned}$$

(for example, use $\mathcal{L}\{\mathrm{e}^{at}\} = 1/(s-a)$ with $a = \mathrm{i}k$). Then

$$\bar{f}(s) = \mathrm{Re}\left(\frac{1}{s-\mathrm{i}k} \times \frac{s+\mathrm{i}k}{s+\mathrm{i}k}\right) = \mathrm{Re}\left(\frac{s+\mathrm{i}k}{s^2+k^2}\right) = \frac{s}{s^2+k^2}.$$

- This way we can build a table of Laplace transforms, which we can use to find inverse Laplace transforms.
- For each function $\bar{f}(s)$ in the table, there is a corresponding function $f(t)$, such that $\bar{f}(s)$ is the Laplace transform of $f(t)$.

Table of Laplace Transforms

	$f(t)$	$\bar{f}(s)$	
1	1	$\frac{1}{s}, \quad s>0$	
2	t	$\frac{1}{s^2}, \quad s>0$	
3	$t^n \quad n=0,1,2,\ldots$	$\frac{n!}{s^{n+1}}, s>0$	
4	e^{at}	$\frac{1}{s-a}, \quad s>a$	
5	$\sin at$	$\frac{a}{s^2+a^2}, \quad s>0$	
6	$\cos at$	$\frac{s}{s^2+a^2}, \quad s>0$	

- $f(t)$ and $\bar{f}(s)$ are often called *Laplace transform pairs*.
- An example of a Laplace transform pair is

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$

- Without such tables, finding inverse Laplace transforms would not be easy!

Example What is $\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)$?

Use the table result

$$\mathcal{L}\left(\sin at\right) = \frac{a}{s^2+a^2}$$

with $a = \sqrt{3}$ to get

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right) = \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right).$$

Example What is $\mathcal{L}^{-1}\left(\frac{1}{s^{10}} - \frac{1}{s^{11}}\right)$?

Use the table result

$$\mathcal{L}\left(t^n\right) = \frac{n!}{s^{n+1}}$$

with $n = 9$ and $n = 10$ to get

$$\mathcal{L}^{-1}\left(\frac{1}{s^{10}} - \frac{1}{s^{11}}\right) = \frac{t^9}{9!} - \frac{t^{10}}{10!}.$$

Example What is $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+5s+6}\right)$?

This one is **not** in the table. First do partial fractions

$$\begin{aligned} \frac{1}{s^2+5s+6} &= \frac{1}{(s+2)(s+3)} \\ &= \frac{A}{s+2}+\frac{B}{s+3} \\ &= \frac{A(s+3)+B(s+2)}{(s+2)(s+3)}. \end{aligned}$$

This works if $A(s+3)+B(s+2)=1$. Choose values

$$\begin{aligned} s=-2 &\quad\Rightarrow\quad A=1 \\ s=-3 &\quad\Rightarrow\quad -B=1\Rightarrow B=-1. \end{aligned}$$

to get

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+5s+6}\right)=\mathcal{L}^{-1}\left(\frac{1}{s+2}-\frac{1}{s+3}\right)=\mathrm{e}^{-2t}-\mathrm{e}^{-3t}.$$

Example: Use Laplace transforms to solve first order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = \mathrm{e}^{2t} \quad \text{with} \quad y(0) = 1. \tag{1}$$

Solution: There are three stages in the process.

(i) First take the Laplace transform of both sides of (1), which gives

$$\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}t}\right\} + \mathcal{L}\left\{y\right\} = \mathcal{L}\left\{\mathrm{e}^{2t}\right\},$$

and using in particular $\mathcal{L}\left(\mathrm{d}y/\mathrm{d}t\right) = s\bar{y}(s) - y(0)$,

$$s\bar{y}(s) - 1 + \bar{y}(s) = \frac{1}{s-2}. \tag{2}$$

(ii) Solve the algebraic equation (2) for $\bar{y}(s)$,

$$
\begin{aligned}
s\bar{y}(s) - 1 + \bar{y}(s) = \frac{1}{s-2} \quad &\Rightarrow \quad (s+1)\bar{y}(s) = 1 + \frac{1}{s-2} \\
&\Rightarrow \quad \bar{y}(s) = \frac{1}{s+1} + \frac{1}{(s+1)(s-2)} \\
&\Rightarrow \quad \bar{y}(s) = \frac{s-1}{(s+1)(s-2)}
\end{aligned}
$$

(iii) Now we ask, what function $y(t)$ has this Laplace transform? Formally we write

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\bar{y}(s)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)(s-2)}\right\}.
\end{aligned}
$$

Here we use *partial fractions* to write

$$\bar{y}(s) = \frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{2}{3(s+1)} + \frac{1}{3(s-2)}.$$

Hence we see from

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

that

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}, \quad \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t},$$

and hence that

$$y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

Remember that this has automatically accounted for *initial conditions*!

Example A second order ODE.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = t \quad \text{with} \quad y(0) = 1, \quad y'(0) = 0. \tag{3}$$

(i) Taking the Laplace transform of both sides of (3) gives

$$s^2 \bar{y}(s) - y'(0) - sy(0) + \bar{y}(s) = \frac{1}{s^2}.$$

Applying the initial conditions leads to

$$(s^2 + 1)\bar{y}(s) = \frac{1}{s^2} + s.$$

(ii) This can now be solved,

$$\bar{y}(s) = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1}.$$

(iii) We want to apply the inverse Laplace transform to

$$\bar{y}(s) = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1}.$$

The second term appears directly in the table of Laplace transforms:

$$\frac{s}{s^2+1} = \mathcal{L}\{\cos t\}$$

The first term can be dealt with using partial fractions,

$$\begin{aligned} \frac{1}{s^2(s^2+1)} &= \frac{1}{s^2} - \frac{1}{s^2+1} \\ &= \mathcal{L}\{t - \sin t\} \end{aligned}$$

(using tables). Then

$$y(t) = t - \sin t + \cos t.$$

We can easily check that y obeys the ODE and the initial conditions,

$$y(0) = \cos 0 = 1, \qquad y'(0) = 1 - \cos(0) = 0.$$

Example: Two coupled ODEs: let x and y satisfy the ODEs

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + x - 3y &= 0 \qquad (4) \\ \frac{\mathrm{d}y}{\mathrm{d}t} + 3x - y &= \mathrm{e}^{-t}, \qquad (5) \end{aligned}$$

with $x(0) = 0$ and $y(0) = 0$.

Solution:

(i) Taking the Laplace transform of equations (4) and (5) gives two simultaneous algebraic equations for $\bar{x}(s)$ and $\bar{y}(s)$:

$$\begin{aligned} s\bar{x} + \bar{x} - 3\bar{y} &= 0, \qquad (6) \\ s\bar{y} + 3\bar{x} - \bar{y} &= \frac{1}{s+1}. \qquad (7) \end{aligned}$$

(ii) Use (6) to eliminate

$$\bar{x} = \frac{3\bar{y}}{s+1}$$

in (7), leading to

$$(s-1)\bar{y} + \frac{9\bar{y}}{s+1} = \frac{1}{s+1}$$

$$\Rightarrow \quad \left((s+1)(s-1)+9\right)\bar{y} = (s^2+8)\bar{y} = 1$$

$$\Rightarrow \quad \bar{y}(s) = \frac{1}{s^2+8}.$$

(iii) Invert using the table,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+8}\right\} = \frac{1}{\sqrt{8}}\sin(\sqrt{8}t).$$

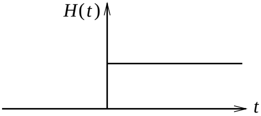
Then x can be found from rearranging equation (5),

$$3x = y - \frac{dy}{dt} + e^{-t} \quad \Rightarrow \quad x(t) = \frac{1}{3\sqrt{8}}\sin(\sqrt{8}t) - \frac{1}{3}\cos(\sqrt{8}t) + \frac{1}{3}e^{-t}.$$

These are two useful functions for modelling eg a sudden pulse or a discontinuous forcing function.

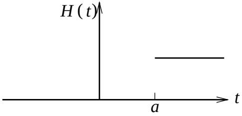
The *Heaviside step function* H is defined by

$$H(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ 1 & \text{for} \quad t \geq 0. \end{cases}$$



Therefore $H(t-a)$ is

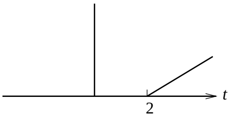
$$H(t-a) = \begin{cases} 0 & \text{for} \quad t < a \\ 1 & \text{for} \quad t \geq a. \end{cases}$$



So $H(t-a)$ describes for example a current that is switched on at $t = a$.

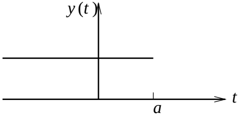
Examples of functions that can be described using H.

(i) $(t-2)H(t-2) = \begin{cases} 0 & \text{for} \quad t < 2 \\ t-2 & \text{for} \quad t \geq 2. \end{cases}$



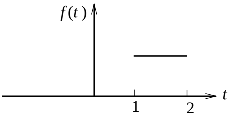
(ii) Let $y(t) = 1 - H(t-a)$. Then

$$y(t) = \begin{cases} 1 & \text{for} \quad t < a \\ 0 & \text{for} \quad t \geq a. \end{cases}$$



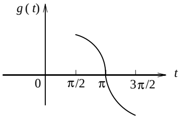
(iii) Let $f(t) = H(t-1) - H(t-2)$. Then

$$f(t) = \begin{cases} 0 & \text{for} \quad t < 1 \\ 1 & \text{for} \quad 1 \leq t < 2 \\ 0 & \text{for} \quad t \geq 2. \end{cases}$$



(iv) Let $g(t) = \sin t \left[H\left(t - \frac{\pi}{2}\right) - H\left(t - \frac{3\pi}{2}\right)\right]$. Then

$$g(t) = \begin{cases} 0 & \text{for} \quad t < \frac{\pi}{2} \\ \sin t & \text{for} \quad \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ 0 & \text{for} \quad t \ge \frac{3\pi}{2}. \end{cases}$$



The Laplace transform of $H(t-a)$ is *Very Useful*:

$$\begin{aligned} \mathcal{L}\{H(t-a)\} &= \int_0^\infty e^{-st} H(t-a)\, dt \\ &= \int_a^\infty e^{-st}\, dt \\ &= \frac{e^{-as}}{s}. \end{aligned}$$

The *Dirac delta function* is defined by the conditions

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0,$$

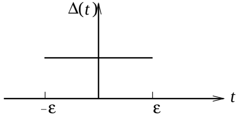
and

$$\int_{-a}^{a} \delta(t)\,\mathrm{d}t = 1 \quad \text{for any } a > 0.$$

The name "delta function" is a bit misleading: it is not a function in the strict mathematical sense. It is something more general, called a *distribution*.

We are more concerned with how to use it. Let us just accept the definition above, and try to obtain a picture of $\delta(t)$ by considering the function

$$\Delta(t) = \begin{cases} \dfrac{1}{2\epsilon} & \text{for} \quad -\epsilon < t < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$



Then

$$\int_{-\infty}^{\infty} \Delta(t)\,\mathrm{d}t = \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}\,\mathrm{d}t = \left[\frac{t}{2\epsilon}\right]_{-\epsilon}^{\epsilon} = 1.$$

We can formally write

$$\delta(t) = \lim_{\epsilon\to 0} \Delta(t).$$

Note that this is not a limit in the strict mathematical sense, as the limit does not exist at $t = 0$. But it becomes a correct limit when we integrate it:

$$\int_{-\infty}^{\infty} \delta(t)f(t)\,\mathrm{d}t = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \Delta(t)f(t)\,\mathrm{d}t,$$

where f is any continous function.

We can use the delta function to represent a quantity which occupies a very small region of space, or exists for an instant of time, for example point force, point charge, impulse.

An important property of the delta function is that

$$\int_b^c f(t)\delta(t-a)\,\mathrm{d}t = f(a),$$

provided the range of integration includes $t = a$, ie $b < a < c$.

The Laplace transform of $\delta(t-a)$ can be found using this result:

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \mathrm{e}^{-st}\delta(t-a)\,\mathrm{d}t = \mathrm{e}^{-sa},$$

provided $a > 0$.

5.5 Some properties of Laplace transforms

The Laplace transform has some properties that are useful for solving ODEs (we have already used some of them without making a big deal about it).

A. **Linearity**

$$\mathcal{L}\{af(t) + b\,g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\},$$

$$\text{eg} \quad \mathcal{L}\left\{e^{2t} + 2e^{-t}\right\} = \frac{1}{s-2} + \frac{2}{s+1},$$

and therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{2}{s+1}\right\} = e^{2t} + 2e^{-t}.$$

B. **First Shifting Theorem**

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\ &= \bar{f}(s-a).\end{aligned}$$

Using the *First Shifting Theorem* with $a = 2$, we get

$$\mathcal{L}\left\{\sin 3t\right\} = \frac{3}{s^2+9}, \quad \text{so} \quad \mathcal{L}\left\{e^{2t}\sin 3t\right\} = \frac{3}{(s-2)^2+9}.$$

Hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{s^2-4s+13}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\} \\ &= e^{2t}\sin 3t.\end{aligned}$$

C. **Second Shifting Theorem**

We can derive the *Second Shifting Theorem* by considering the transform of $f(t-a)H(t-a)$:

$$\begin{aligned}\mathcal{L}\{f(t-a)H(t-a)\} &= \int_0^\infty \mathrm{e}^{-st} f(t-a)H(t-a)\,\mathrm{d}t \\ &= \int_a^\infty \mathrm{e}^{-st} f(t-a)\,\mathrm{d}t.\end{aligned}$$

Substituting $t = u + a$ gives

$$= \int_0^\infty \mathrm{e}^{-s(u+a)} f(u)\,\mathrm{d}u = \mathrm{e}^{-as}\int_0^\infty \mathrm{e}^{-su} f(u)\,\mathrm{d}u = \mathrm{e}^{-as}\bar{f}(s).$$

Therefore

$$\mathcal{L}\{f(t-a)H(t-a)\} = \mathrm{e}^{-as}\bar{f}(s).$$

Example: Solve

$$\frac{d^2y}{dt^2} + 9y = 10\,\delta(t-2)$$

with $y(0) = 0$ and $y'(0) = 1$.

Solution: Taking the Laplace transform of both sides gives

$$(s^2+9)\bar{y} - 1 = 10e^{-2s} \quad \Rightarrow \quad \bar{y} = \frac{1}{s^2+9} + \frac{10e^{-2s}}{s^2+9}.$$

Now

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t),$$

and from the second shifting theorem we get

$$\mathcal{L}^{-1}\left\{\frac{10e^{-2s}}{s^2+9}\right\} = \frac{10}{3}\sin\left(3(t-2)\right)H(t-2),$$

Hence

$$y(t) = \frac{1}{3}\sin 3t + \frac{10}{3}\sin 3(t-2)H(t-2).$$

D. **The Convolution Theorem**

The *convolution* $f \star g$ of two functions $f(t)$ and $g(t)$ defined for $t > 0$ is

$$(f \star g)(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u.$$

It can be shown, using double integrals, that the Laplace transform of the convolution is

$$\mathcal{L}\{(f \star g)(t)\} = \bar{f}(s)\bar{g}(s),$$

or alternatively

$$\mathcal{L}^{-1}\left\{\bar{f}(s)\,\bar{g}(s)\right\} = (f \star g)(t) = \int_0^t f(u)g(t-u)\mathrm{d}u.$$

This result is called the *Convolution Theorem*. It is sometimes useful for inverting Laplace transforms.

Example: Find

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2(s^2+k^2)}\right\}.$$

Solution: Let

$$\bar{f}(s) = \frac{1}{s^2} \quad \text{and} \quad \bar{g}(s) = \frac{k}{s^2+k^2}.$$

Then

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2(s^2+k^2)}\right\} = \mathcal{L}^{-1}\left\{\bar{f}(s)\,\bar{g}(s)\right\}.$$

From the table of Laplace transforms we know that

$$f(t) = t \quad \text{and} \quad g(t) = \sin(kt).$$

Then using the Convolution Theorem we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} &= \int_0^t f(u)g(t-u)\,\mathrm{d}u \\ &= \int_0^t u\sin(k(t-u))\,\mathrm{d}u.\end{aligned}$$

We can work out this convolution integral explicitly using integration by parts:

$$
\begin{aligned}
\int_0^t u \sin(k(t-u))\,\mathrm{d}u &= \left[-u\frac{\cos(k(t-u))}{(-k)}\right]_0^t - \int_0^t \frac{-\cos(k(t-u))}{(-k)}\,\mathrm{d}u \\
&= \frac{t}{k} - \frac{1}{k}\left[\frac{\sin(k(t-u))}{(-k)}\right]_0^t \\
&= \frac{1}{k^2}\left(kt - \sin(kt)\right).
\end{aligned}
$$

Therefore we have shown that

$$
\mathcal{L}^{-1}\left\{\frac{k}{s^2(s^2+k^2)}\right\} = \frac{1}{k^2}\left(kt - \sin(kt)\right).
$$

E. **The Final Value Theorem**

This says that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \left[s\bar{f}(s) \right]$$

provided $\lim_{t\to\infty} f(t)$ exists.

This theorem, which is useful in Control Theory, enables the long-time behaviour of a function to be determined from its Laplace transform without the need to find the complete solution.

This means that the existence, or non-existence, of the limit of $f(t)$ as $t \to \infty$ can be determined by inspection of $\bar{f}(s)$.

ODEs involving the Heavyside function or the delta function are best solved using Laplace transforms.

Example: An oscillator, initially at rest, has constant forcing that is switched off at $t = 2$, ie

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = 1 - H(t-2)$$

with $y(0) = 0$ and $y'(0) = 0$.

Solution: Taking the Laplace transform, we get

$$s^2\bar{y} + \bar{y} = \frac{1}{s} - \frac{\mathrm{e}^{-2s}}{s} \quad \Rightarrow \quad \bar{y} = \frac{1}{s(s^2+1)}(1 - \mathrm{e}^{-2s}).$$

Using partial fractions reveals

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1} \quad \text{where} \quad A = 1, \quad B = -1, \quad C = 0.$$

Therefore

$$\bar{y} = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)(1 - \mathrm{e}^{-2s}).$$

Now invert $\bar{y}(s)$. The factor in brackets can be inverted using the tables:

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t.$$

The term

$$\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\mathrm{e}^{-2s}$$

can then be inverted using the second shifting theorem, giving

$$\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\mathrm{e}^{-2s}\right\} = [1 - \cos(t-2)]\,H(t-2).$$

The full solution is

$$y(t) = 1 - \cos t - [1 - \cos(t-2)]\, H(t-2).$$

Since

$$H(t-2) = \begin{cases} 0 & \text{for } t < 2 \\ 1 & \text{for } t \geq 2, \end{cases}$$

the solution can be written as

$$y(t) = \begin{cases} 1 - \cos t & \text{for} \quad 0 < t < 2, \\ -\cos t + \cos(t-2) & \text{for} \quad t \geq 2. \end{cases}$$

Again, you can check that y obeys the ODE and the initial conditions.