4. Let the function $\varphi(x, y)$ satisfy the partial differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + 2\frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{1}$$

in the square domain

$$0 < x < 1 \quad \text{and} \quad 0 < y < 1.$$

(a) Show that a separation of variables substitution $\varphi(x, y) = X(x)Y(y)$ leads to an equation for $X(x)$ of the form

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda X = 0, \tag{2}$$

where λ is a constant, and find the corresponding equation for $Y(y)$.

[5 marks]

(b) The boundary conditions

$$\varphi(x = 0, y) = 0 \qquad \text{and} \qquad \varphi(x = 1, y) = 0, \qquad \text{for all} \quad 0 < y < 1 \tag{3}$$

are imposed. Find all solutions of (2) consistent with these and find the corresponding most general solutions for $Y(y)$.

[8 marks]

(c) Solve (1) if the boundary conditions

$$\varphi(x, y = 0) = 0 \qquad \text{and} \qquad \varphi(x, y = 1) = \sin(\pi x) + 2\sin(2\pi x),$$

are imposed, in addition to those in (3).
Hint: the boundary condition along $y = 1$ is in the form of a Fourier series where all but two Fourier coefficients are zero.

[7 marks]