Q4(a) $\varphi_{xx}+2\varphi_{yy}=0$ with $\varphi = X(x)Y(y)$ implies

$$X''Y + 2XY'' = 0$$

$$\Rightarrow X''/X + 2Y''/Y = 0$$

$$\Rightarrow \underbrace{X''/X}_{\text{fn of } x} = \underbrace{-2Y''/Y}_{\text{fn of } y} = -\lambda \text{ (say)}$$

$$\Rightarrow X'' + \lambda X = 0 \text{ and } Y'' - \tfrac{1}{2}\lambda Y = 0$$

(b) $X'' + \lambda X = 0$ has the general solutions

(i) $\lambda = -k^2 < 0 \Rightarrow X = Ae^{kx} + Be^{-kx}$
(ii) $\lambda = 0 \Rightarrow X = A + Bx$
(iii) $\lambda = k^2 > 0 \Rightarrow X = A\cos kx + B\sin kx$

The boundary conditions imply $X(0) = 0 = X(1)$.
In cases (i) and (ii) these conditions imply $A=B=0$.
In case (iii)

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin kx$$

$$X(1) = 0 \Rightarrow \sin k = 0 \Rightarrow k = n\pi \quad n = 1,2,3\ldots$$

if $B \neq 0$, which is required for nontrivial solution

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Corresponding values of λ: $\lambda = k^2 = n^2\pi^2 \quad n = 1, 2, 3, \ldots$

Corresponding equations for Y: $Y'' - \frac{n^2\pi^2}{2} Y = 0$

$$\Rightarrow Y(y) = C e^{\frac{n\pi}{\sqrt{2}} y} + D e^{-\frac{n\pi}{\sqrt{2}} y}$$

(c) General solution subject to ic's in (b):

$$\phi(x,y) = \sum_{n=1}^{\infty} \left(C_n e^{\frac{n\pi}{\sqrt{2}} y} + D_n e^{-\frac{n\pi}{\sqrt{2}} y} \right) \sin n\pi x$$

$\phi(x,0) = 0$ for all $x \Rightarrow D_n = -C_n$

$$\Rightarrow \phi(x,y) = \sum_{n=1}^{\infty} 2C_n \sinh\left(\frac{n\pi}{\sqrt{2}} y \right) \sin n\pi x$$

$$\phi(x,1) = \sum_{n=1}^{\infty} 2C_n \sinh\left(n\pi/\sqrt{2} \right) \sin n\pi x$$

$$= \sin \pi x + 2 \sin 2\pi x$$

$$\Rightarrow 2C_1 \sinh \frac{\pi}{\sqrt{2}} = 1, \quad 2C_2 \sinh \frac{2\pi}{\sqrt{2}} = 2$$

and all other $C_n = 0$

$$\Rightarrow \phi(x,y) = \frac{\sinh\left(\frac{\pi}{\sqrt{2}} y \right)}{\sinh(\pi/\sqrt{2})} \sin \pi x$$

$$+ 2 \frac{\sinh(\sqrt{2}\pi y)}{\sinh(\sqrt{2}\pi)} \sin 2\pi x$$