Problem Sheet 1 – Solutions

1) a) Guess $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\Rightarrow y'' - y = m^2 e^{mx} - e^{mx} = e^{mx}(m^2 - 1) = 0$

$e^{mx} \neq 0 \Rightarrow m^2 - 1 = 0 \Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

Auxilliary equation

$\Rightarrow e^{x}$ & e^{-x} are both solutions

$\Rightarrow$ General solution is $\boxed{y = Ae^{x} + Be^{-x}}$

b) $y'' + y = 0 \Rightarrow$ Aux. eqn is $m^2 + 1 = 0$

$\Rightarrow m^2 = -1 \Rightarrow m = \pm i$

$\Rightarrow \boxed{y = A\cos x + B\sin x}$

c) $y'' + 8y' + 15y = 0 \Rightarrow$ Aux. eqn is $m^2 + 8m + 15 = 0$

$\Rightarrow (m+5)(m+3) = 0$

$\Rightarrow m = -5$ or -3

$\Rightarrow \boxed{y = Ae^{-5x} + Be^{-3x}}$

d) $y'' + 6y' + 13y = 0 \Rightarrow$ Aux. eqn. is $m^2 + 6m + 13 = 0$

$\Rightarrow m^2 + 6m + 9 + 4 = 0$

$\Rightarrow (m+3)^2 = -4 \Rightarrow m + 3 = \pm 2i$

$\Rightarrow \quad m = -3 \pm 2i$

$\Rightarrow \quad \boxed{y = e^{-3x}(A\cos 2x + B\sin 2x)}$

e) $y'' - 6y' + 9y = 0 \Rightarrow$ Aux. eqn is $m^2 - 6m + 9 = 0$

$\Rightarrow \quad (m-3)^2 = 0 \quad \Rightarrow \quad m = 3$, repeated root

$\Rightarrow \quad \boxed{y = (A + Bx)e^{3x}}$

2) a) $y'' + 4y' - 12y = 14e^x$

$\Rightarrow$ Aux. eqn is $m^2 + 4m - 12 = 0$

$\Rightarrow (m+6)(m-2) = 0$

$\Rightarrow m = -6$ or 2

$\Rightarrow$ Complementary function is $y_{CF} = Ae^{-6x} + Be^{2x}$

Try $y_{PI} = ke^x \Rightarrow y_{PI}'' + 4y_{PI}' - 12y_{PI} = ke^x + 4ke^x - 12ke^x$

$= -7ke^x = 14e^x$

$\Rightarrow k = -2 \Rightarrow y_{PI} = -2e^x$

$\Rightarrow$ General solution is $\boxed{y_{CF} + y_{PI} = Ae^{-6x} + Be^{2x} - 2e^x}$

b) $y'' + 4y' + 3y = 8\cos x - 6\sin x$

$\Rightarrow$ Aux. eqn is $m^2 + 4m + 3 = 0$

$\Rightarrow (m+3)(m+1) = 0 \Rightarrow m = -1$ or -3

$\Rightarrow y_{CF} = Ae^{-x} + Be^{-3x}$

Try $y_{PI} = a\cos x + b\sin x$

$y'_{PI} = -a\sin x + b\cos x$

$y''_{PI} = -a\cos x - b\sin x$

$\Rightarrow y''_{PI} + 4y'_{PI} + 3y_{PI} = -a\cos x - b\sin x$
$+4(-a\sin x + b\cos x)$
$+3(a\cos x + b\sin x)$

$= (2b - 4a)\sin x + (2a + 4b)\cos x$

$= -6\sin x + 8\cos x$

$\Rightarrow 2b - 4a = -6 \Rightarrow b - 2a = -3$

$4b + 2a = 8$

add $\Rightarrow$ $5b = 5 \Rightarrow b = 1$

$\Rightarrow 1 - 2a = -3 \Rightarrow a = 2$

$\Rightarrow y_{PI} = 2\cos x + \sin x$

$\Rightarrow \boxed{y = Ae^{-x} + Be^{-3x} + 2\cos x + \sin x}$

c) Aux. eqn. is $m^2 - 9 = 0 \Rightarrow m = \pm 3$

$\Rightarrow y_{CF} = Ae^{3x} + Be^{-3x}$

Guess $y_{PI} = kxe^{3x} \Rightarrow y'_{PI} = k(e^{3x} + 3xe^{3x})$

$\Rightarrow y''_{PI} = k(3e^{3x} + 3e^{3x} + 9xe^{3x}) = k(6e^{3x} + 9xe^{3x})$

$\Rightarrow y''_{PI} - 9y_{PI} = k(6e^{3x} + \cancel{9xe^{3x}}) - \cancel{9kxe^{3x}} = 6ke^{3x} = 42e^{3x}$

$\Rightarrow k = 7 \Rightarrow y_{PI} = 7xe^{3x}$

$$\Rightarrow \boxed{y = Ae^{3x} + Be^{-3x} + 7xe^{3x}}$$

3) a) (see 2)a))

$y = Ae^{-6x} + Be^{2x} - 2e^{x}$

$y' = -6Ae^{-6x} + 2Be^{2x} - 2e^{x}$

$y(0) = A + B - 2 = 1 \Rightarrow A + B = 3$

$y'(0) = -6A + 2B - 2 = -4 \Rightarrow -6A + 2B = -2$

$\Rightarrow -3A + B = -1$

$A + B = 3$

subtract $\quad -4A = -4$

$\Rightarrow A = 1,\ B = 2$

$$\Rightarrow \boxed{y = e^{-6x} + 2e^{2x} - 2e^{x}}$$

b) $y_{cf} = Ae^{-6x} + Be^{2x}$

$y_{PI} = k \Rightarrow y_{PI}'' + 4y_{PI}' - 12y_{PI} = -12k = 3$

$\Rightarrow k = -1/4$

$\Rightarrow y = Ae^{-6x} + Be^{2x} - 1/4$

$y(0) = A + B - 1/4 = 1 \Rightarrow A + B = 5/4$

$y \to$ constant as $x \to \infty$ $\Rightarrow$ $B = 0$ $\Rightarrow A = 5/4$

$$\Rightarrow \boxed{y = \tfrac{5}{4}e^{-6x} - \tfrac{1}{4}}$$

4) $y' + y - z = e^t$ | $\Rightarrow z = y' + y - e^t$

$z' - y + z = e^t$ | $\Rightarrow z' = y'' + y' - e^t$

$$\Rightarrow y'' + y' - e^t \cancel{-y} + y' \cancel{+y} - e^t = e^t$$

$$\Rightarrow y'' + 2y' = 3e^t$$

Aux eqn is $m^2 + 2m = 0 \Rightarrow m(m+2) = 0$

$\Rightarrow m = 0$ or -2

$$\Rightarrow y_{cf} = A + Be^{-2t}$$

Try $y_{PI} = ke^t \Rightarrow y_{PI}'' + 2y_{PI}' = ke^t + 2ke^t = 3e^t$

$\Rightarrow k = 1$

$$\Rightarrow \boxed{y = A + Be^{-2t} + e^t}$$

$$z = y' + y - e^t = -2Be^{-2t} + e^t + A + Be^{-2t} \cancel{+e^t} \cancel{-e^t}$$

$$\boxed{z = A - Be^{-2t} + e^t}$$

5) a) Aux. eqn $m^2 - 2m + 5 = 0$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$\Rightarrow y_{cf} = e^x (A \cos 2x + B \sin 2x)$$

Try $y_{PI} = ax + b$

$$\Rightarrow \frac{d^2 y_{PI}}{dx^2} - 2\frac{dy_{PI}}{dx} + 5y_{PI} = -2a + 5(ax + b)$$

$$= 5ax + 5b - 2a = 5x + 3$$

$$\Rightarrow a = 1 \quad \& \quad 5b - 2 = 3 \Rightarrow b = 1$$

$$\Rightarrow y_{PI} = x + 1$$

$$\Rightarrow \boxed{y = e^x (A \cos 2x + B \sin 2x) + x + 1}$$

b) Aux. eqn is $m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0 \Rightarrow m = -3$, repeated root

$$\Rightarrow y_{CF} = (A + Bt)e^{-3t}$$

Try $y_{PI} = ae^t + b$

$$\Rightarrow \frac{d^2 y_{PI}}{dt^2} + 6\frac{dy_{PI}}{dt} + 9y_{PI} = ae^t + 6ae^t + 9(ae^t + b)$$

$$= 16ae^t + 9b = 32e^t + 9$$

$$\Rightarrow a = 2, \ b = 1$$

$$\Rightarrow y_{PI} = 2e^{t} + 1$$

$$\Rightarrow \boxed{y = (A+Bt)e^{-3t} + 2e^{t} + 1}$$

b) $y = -x - \frac{dx}{dt}$, $\frac{dy}{dt} = -\frac{dx}{dt} - \frac{d^2x}{dt^2}$

$$\Rightarrow \frac{dy}{dt} + 3x - y = -\cancel{\frac{dx}{dt}} - \frac{d^2x}{dt^2} + 3x + x + \cancel{\frac{dx}{dt}} = \sin t$$

$$\Rightarrow -\frac{d^2x}{dt^2} + 4x = \sin t$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} - 4x = -\sin t}$$

Aux. eqn is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

$$\Rightarrow x_{cf} = Ae^{2t} + Be^{-2t}$$

Try $x_{PI} = k\sin t$

$$\Rightarrow \frac{d^2x_{PI}}{dt^2} - 4x_{PI} = -k\sin t - 4k\sin t = -\sin t$$

$$\Rightarrow -5k = -1 \Rightarrow k = 1/5$$

$$\Rightarrow \boxed{x = Ae^{2t} + Be^{-2t} + 1/5 \sin t}$$

$$y = -x - \frac{dx}{dt} = -Ae^{2t} - Be^{-2t} - 1/5\sin t - \left(2Ae^{2t} - 2Be^{-2t} + 1/5\cos t\right)$$

$$\Rightarrow \boxed{y = -3Ae^{2t} + Be^{-2t} - 1/5\sin t - 1/5\cos t}$$

7) a) Aux. eqn is $m^2 + m - 2 = 0$

$$\Rightarrow (m+2)(m-1) = 0$$

$$\Rightarrow m = -2 \text{ or } 1$$

$$\Rightarrow y_{CF} = Ae^{-2x} + Be^{x}$$

Try $y_{PI} = (ax+b)e^{2x}$

$$\Rightarrow y'_{PI} = 2(ax+b)e^{2x} + ae^{2x} = (2ax + 2b + a)e^{2x}$$

$$\Rightarrow y''_{PI} = 2(2ax + 2b + a)e^{2x} + 2ae^{2x}$$

$$= (4ax + 4b + 4a)e^{2x}$$

$$\Rightarrow y''_{PI} + y'_{PI} - 2y_{PI} = (4ax + 4b + 4a)e^{2x} + (2ax + 2b + a)e^{2x} - 2(ax+b)e^{2x} = xe^{2x}$$

$$\Rightarrow 4ax + 4b + 5a = x$$

$$\Rightarrow a = \tfrac{1}{4} \ \& \ 4b + \tfrac{5}{4} = 0 \Rightarrow b = -\tfrac{5}{16}$$

$$\Rightarrow y_{PI} = \left(\tfrac{1}{4}x - \tfrac{5}{16}\right)e^{2x}$$

$$\Rightarrow \boxed{y = Ae^{-2x} + Be^{x} + \left(\tfrac{1}{4}x - \tfrac{5}{16}\right)e^{2x}}$$

b) Try $y_{PI} = (ax^2 + bx + c)e^{2x}$

$$\Rightarrow y'_{PI} = 2(ax^2 + bx + c)e^{2x} + (2ax + b)e^{2x}$$

$$y'_{PI} = (2ax^2 + (2a+2b)x + b + 2c)e^{2x}$$

$$y''_{PI} = 2(2ax^2 + (2a+2b)x + b + 2c)e^{2x}$$
$$+ (4ax + (2a+2b))e^{2x}$$

$$= (4ax^2 + (8a+4b)x + 2a + 4b + 4c)e^{2x}$$

$$\Rightarrow y''_{PI} + y'_{PI} - 2y_{PI}$$
$$= (4ax^2 + (8a+4b)x + 2a + 4b + 4c)e^{2x}$$
$$+ (2ax^2 + (2a+2b)x + b + 2c)e^{2x}$$
$$-2(ax^2 + bx + c)e^{2x}$$

$$= (4ax^2 + (10a+4b)x + 2a + 5b + 4c)e^{2x}$$
$$= x^2e^{2x}$$

$$\Rightarrow a = 1/4 \quad \& \quad 10/4 + 4b = 0 \Rightarrow b = -5/8$$

$$\& \quad \frac{1}{2} - \frac{25}{8} + 4c = 0$$

$$\Rightarrow 4c = 21/8 \Rightarrow c = 21/32$$

$$\Rightarrow \boxed{y = Ae^{-2x} + Be^{x} + \left(\frac{1}{4}x^2 - \frac{5}{8}x + \frac{21}{32}\right)e^{2x}}$$