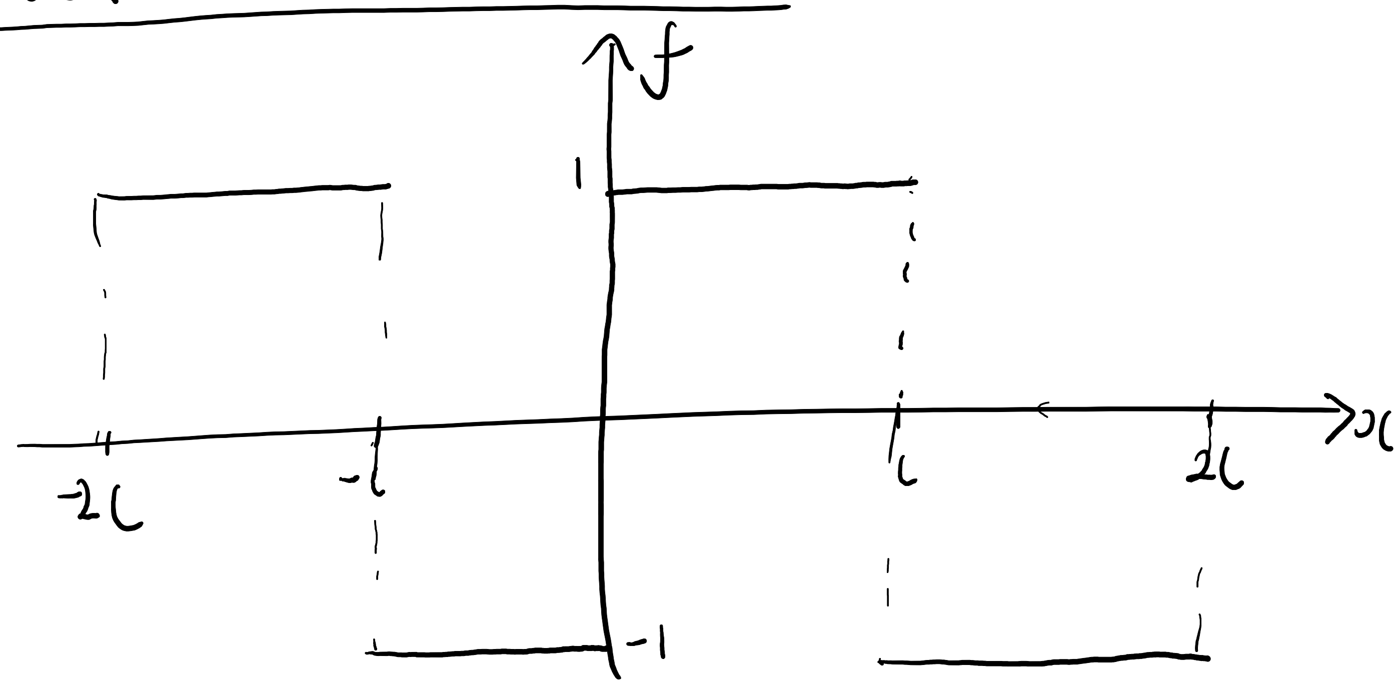
Problem Sheet 2 – Solutions

1)



This is an odd function, i.e. $f(x) = -f(-x)$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_{0}^{l} \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{\cancel{l}}\left[-\frac{\cancel{l}}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= -\frac{2}{n\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{-2}{n\pi}\left((-1)^n - 1\right) = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

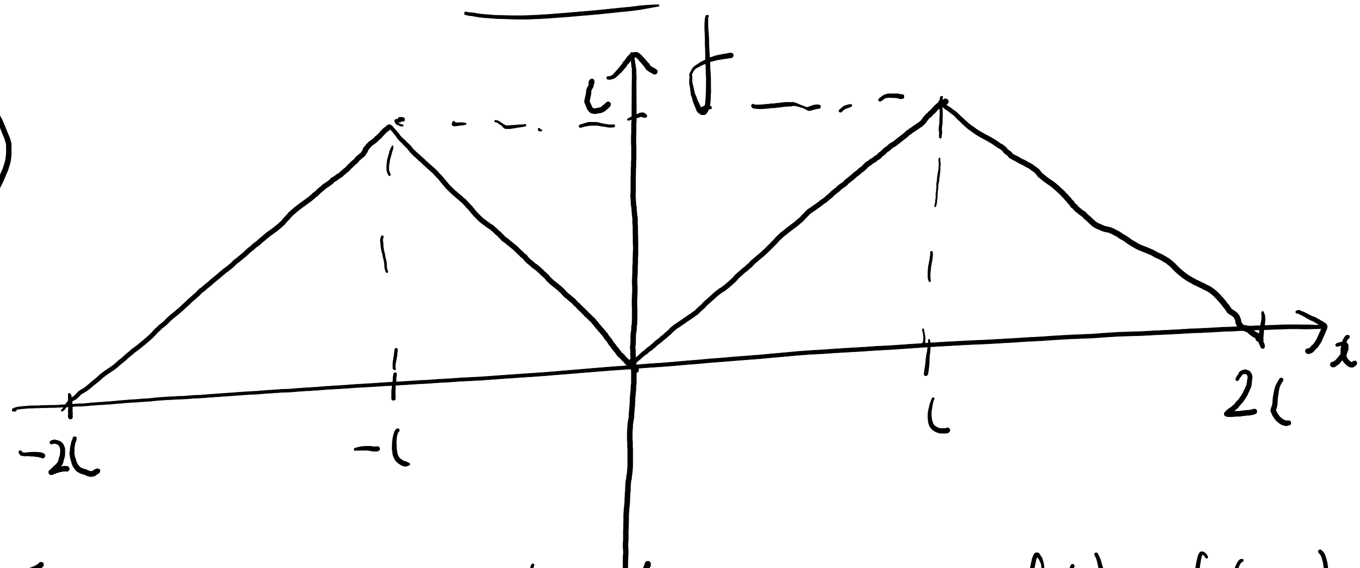
$$\Rightarrow f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(1-(-1)^n\right)\frac{1}{n}\,\sin\left(\frac{n\pi x}{L}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

$$\Rightarrow f(x) = \frac{4}{\pi}\sum_{n,\,\text{odd}}\frac{1}{n}\,\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\,\sin\left(\frac{m\pi x}{L}\right)}$$

2)



This an even function, i.e. $f(x) = f(-x)$

Series converges to $f(0) = 0$ & $f(L) = L$ (a continuous function)

$$\Rightarrow f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x \cos\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow a_0 = \frac{2}{l}\int_0^l x \, dx = \frac{2}{l}\left[\frac{1}{2}x^2\right]_0^l$$

$$= \frac{2}{l}\cdot\frac{1}{2}l^2 = l$$

for $n>0$ $\quad a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\left[x\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{2}{l}\int_0^l \frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left(l\cdot\frac{l}{n\pi}\underbrace{\sin n\pi}_{=0}\right) - \frac{2}{l}\left[\frac{l}{n\pi}\cdot -\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}(\cos n\pi - 1) = \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$a_n = \begin{cases} 0 & , \quad n \text{ even} \\ -4l/n^2\pi^2 & , \quad n \text{ odd} \end{cases}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{2}l - \sum_{n,\text{odd}} \frac{4l}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)}$$

Try $x = 0$, where $f(0) = 0$.

$$\Rightarrow 0 = \frac{1}{2}l - \sum_{n,\text{odd}} \frac{4l}{n^2\pi^2}$$

$$\Rightarrow \sum_{n,\text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8} = 1 + 1/9 + 1/25 + \ldots$$

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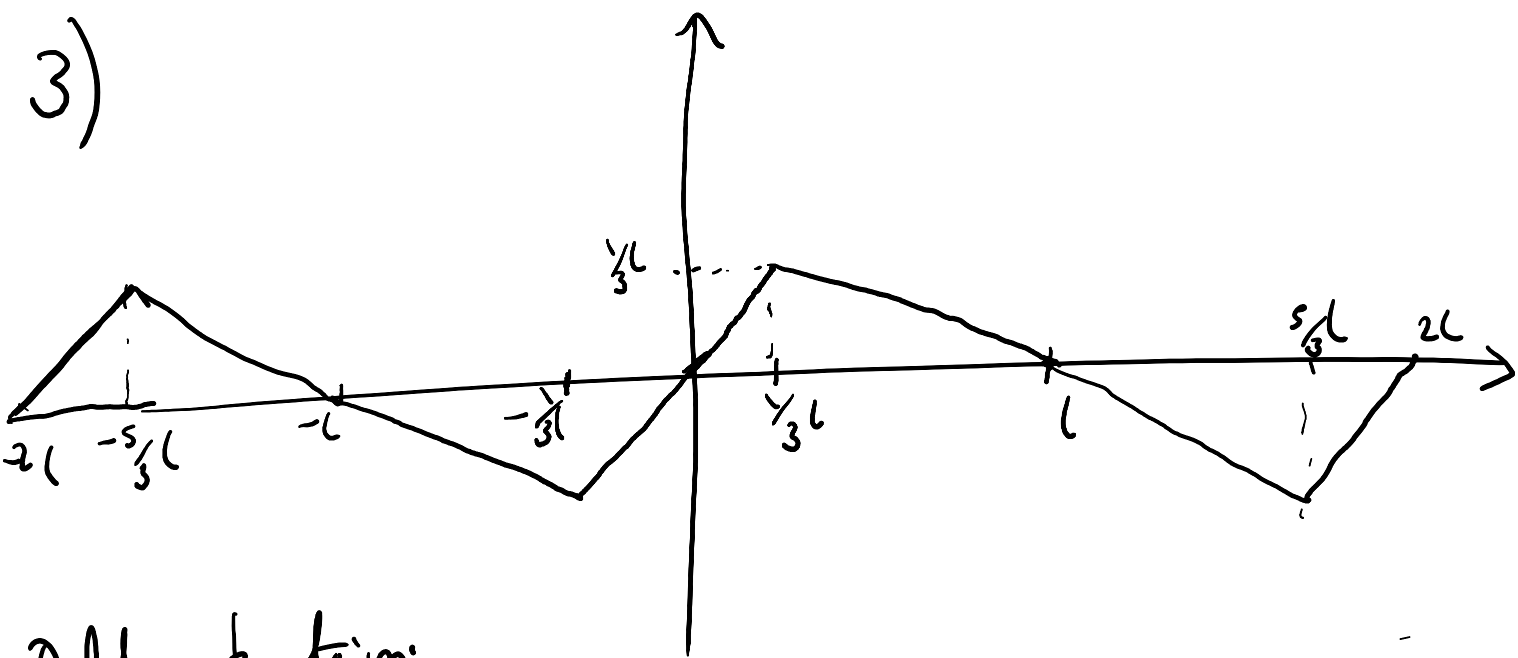
3)



Odd function

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L}\int_0^{\frac{1}{3}L} x \sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{\frac{1}{3}L}^{L} \frac{1}{2}(L-x) \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{1}{3}L} + \frac{2}{L}\int_0^{\frac{1}{3}L}\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$+ \frac{2}{L}\left[-\frac{1}{2}(L-x)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{\frac{1}{3}L}^{L} - \frac{2}{L}\int_{\frac{1}{3}L}^{L}\frac{1}{2}\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx$$

In the video I forgot this minus sign! Sorry

$$= \cancel{\frac{2}{L}\cdot -\frac{1}{3}L\cdot\frac{L}{n\pi}\cdot\cos\left(\frac{1}{3}n\pi\right)} + \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{1}{3}L}$$

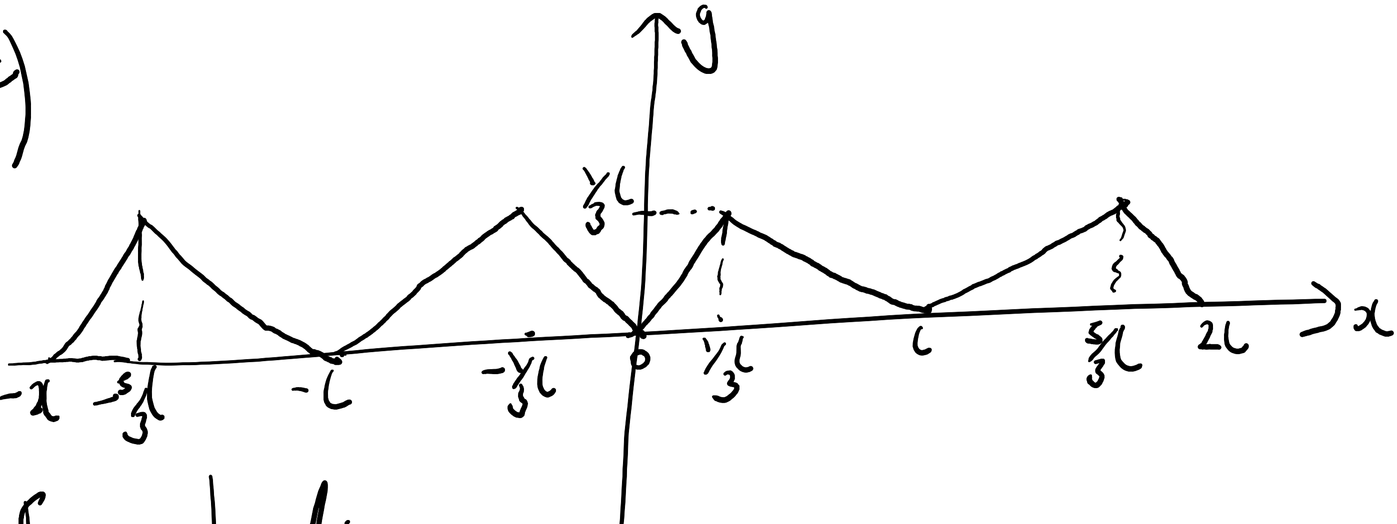
$$+ \cancel{\frac{2}{L}\cdot\frac{1}{3}L\cdot\frac{L}{n\pi}\cos\left(\frac{1}{3}n\pi\right)} - \frac{1}{L}\left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_{\frac{1}{3}L}^{L}$$

$$= \frac{2L}{n^2\pi^2}\sin\left(\frac{1}{3}n\pi\right) + \frac{L}{n^2\pi^2}\sin\left(\frac{1}{3}n\pi\right)$$

$$= \frac{3L}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty} \frac{3l}{n^2\pi^2} \sin\left(\frac{n\pi}{3}\right) \sin\left(\frac{n\pi x}{l}\right)}$$

4)



Even function

$$\Rightarrow g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_n = \frac{2}{l}\int_0^l g(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$a_0 = \frac{2}{l}\int_0^l g(x)\, dx = \frac{2}{l}\int_0^{\frac{1}{3}l} x\, dx + \frac{2}{l}\int_{\frac{1}{3}l}^{l} \frac{1}{2}(l-x)\, dx$$

$$= \frac{2}{l}\left[\frac{1}{2}x^2\right]_0^{\frac{1}{3}l} + \frac{2}{l}\left[\frac{1}{2}lx - \frac{1}{4}x^2\right]_{\frac{1}{3}l}^{l}$$

$$= \frac{1}{l}\left(\frac{1}{3}l\right)^2 + \frac{2}{l}\left[\frac{1}{2}l^2 - \frac{1}{4}l^2 - \frac{1}{6}l^2 + \frac{1}{36}l^2\right]$$

$$= \frac{1}{9}l + \frac{2}{l}\,\frac{1}{36}l^2\left[18 - 9 - 6 + 1\right]$$

$$= \frac{1}{9}l + \frac{2}{9}l = \frac{1}{3}l$$

For $n>0$

$$a_n = \frac{2}{l}\int_0^{\frac{1}{3}l} x\cos\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\int_{\frac{1}{3}l}^{l} \frac{1}{2}(l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^{\frac{1}{3}l} - \frac{2}{l}\int_0^{\frac{1}{3}l}\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$+\frac{2}{l}\left[\frac{1}{2}(l-x)\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_{\frac{1}{3}l}^{l} + \frac{1}{l}\int_{\frac{1}{3}l}^{l}\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \cancel{\frac{2l}{3n\pi}\sin\left(\frac{n\pi}{3}\right)} - \frac{2}{n\pi}\left[-\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^{\frac{1}{3}l}$$

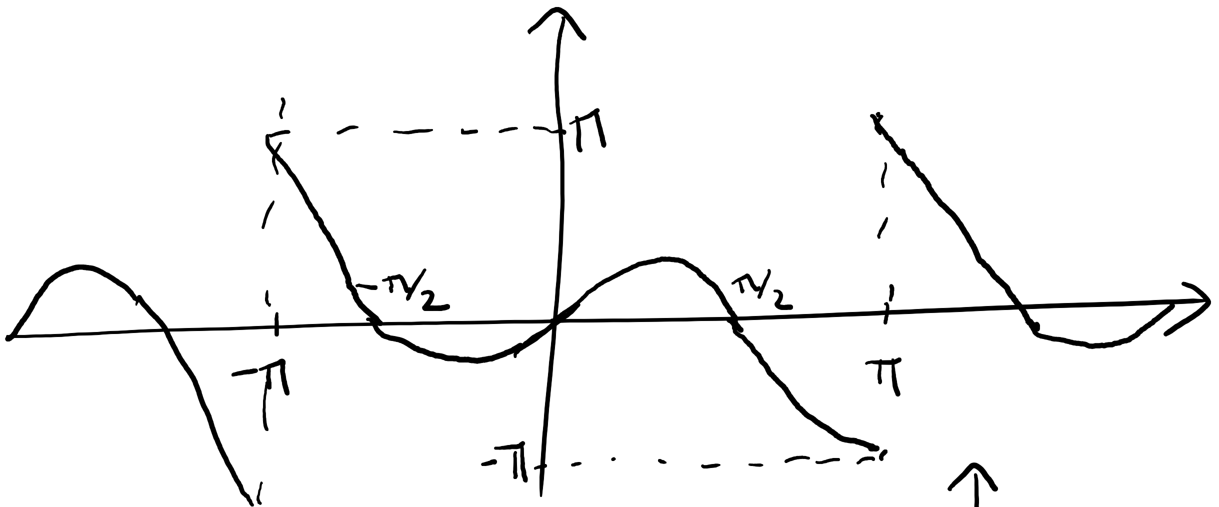
$$- \cancel{\frac{2l}{3n\pi}\sin\left(\frac{n\pi}{3}\right)} + \frac{1}{n\pi}\left[-\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_{\frac{1}{3}l}^{l}$$

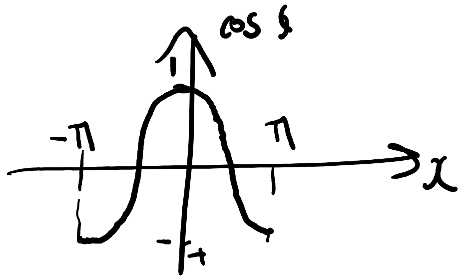
$$= \frac{2l}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{3}\right)-1\right) - \frac{l}{n^2\pi^2}\left((-1)^n - \cos\left(\frac{n\pi}{3}\right)\right)$$

$$= \frac{l}{n^2\pi^2}\left(3\cos\left(\frac{n\pi}{3}\right) - 2 - (-1)^n\right)$$

$$\Rightarrow \boxed{g(x) = \frac{1}{6}l + \sum_{n=1}^{\infty}\frac{l}{n^2\pi^2}\left(3\cos\left(\frac{n\pi}{3}\right) - 2 - (-1)^n\right)\cos\left(\frac{n\pi x}{l}\right)}$$

5)





Discontinuous at multiples of π.

Odd function.

a) Fourier series converges to $\frac{1}{2}(\pi + (-\pi)) = 0$.

b) Period $2\pi \Rightarrow l = \pi$ in Fourier formula.

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos x \sin nx \, dx$$

Hint says

$$\sin(n+1)x = \sin nx \cos x + \cos nx \sin x$$
$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$$

$$\Rightarrow \sin nx \cos x = \frac{1}{2}\left\{\sin(n+1)x + \sin(n-1)x\right\}$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \frac{1}{2}\left\{\sin(n+1)x + \sin(n-1)x\right\} dx$$

$$= \frac{1}{\pi}\left[x \cdot \left\{\frac{-1}{n+1}\cos(n+1)x - \frac{1}{n-1}\cos(n-1)x\right\}\right]_0^{\pi}$$

$$+ \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right) dx$$

$$= \frac{-1}{n+1}\cos(n+1)\pi + \frac{-1}{n-1}\cos(n-1)\pi$$

$$+ \frac{1}{\pi}\underbrace{\left[\frac{1}{(n+1)^2}\sin(n+1)x + \frac{1}{(n-1)^2}\sin(n-1)x\right]_0^{\pi}}_{=0}$$

$$= \frac{-1}{n+1}(-1)^{n+1} - \frac{1}{n-1}(-1)^{n-1} \quad \left(\begin{array}{c}\text{assuming} \\ n \neq 1\end{array}\right)$$

$$= -(-1)^n\left(\frac{-1}{n+1} - \frac{1}{n-1}\right)$$

$$= (-1)^n\left(\frac{1}{n+1} + \frac{1}{n-1}\right) = \frac{(-1)^n 2n}{(n^2-1)}$$

$$\Rightarrow b_n = \frac{(-1)^n 2n}{n^2-1} \qquad n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^\pi x \sin 2x \quad dx$$

$$= \frac{1}{\pi}\left[-\tfrac{1}{2}x \cos 2x\right]_0^\pi + \frac{1}{2\pi}\int_0^\pi \cos 2x \, dx$$

$$= -\tfrac{1}{2} + \frac{1}{2\pi}\left[\tfrac{1}{2}\underbrace{\cancel{\sin 2x}}_{=0}\right]_0^\pi = -\tfrac{1}{2}$$

$$\Rightarrow \boxed{f(x) = -\tfrac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{(-1)^n 2n}{n^2-1} \sin nx}$$

c) $f(x) = 0$ at $x = \pi$,