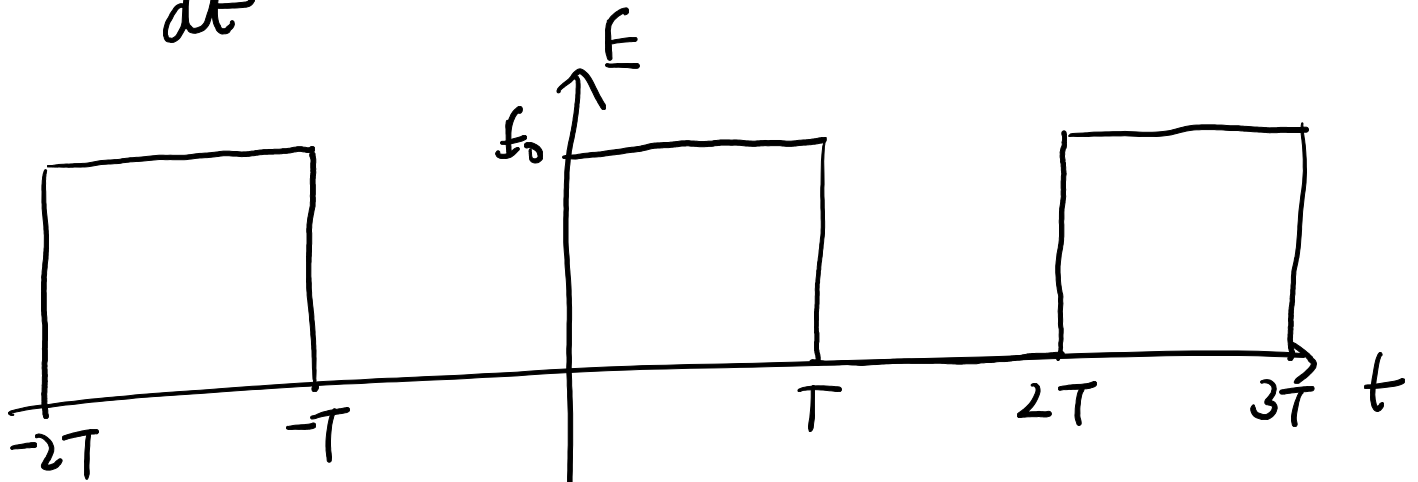


Problem Sheet 3 - Solutions

1) $L \frac{dI}{dt} + RI = E$



Weiter odd nor even. period $2T$.

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right)$$

$$a_n = \frac{1}{T} \int_{-T}^T f(t) \cos \frac{n\pi t}{T} dt, \quad b_n = \frac{1}{T} \int_{-T}^T f(t) \sin \frac{n\pi t}{T} dt$$

$$a_0 = \frac{1}{T} \int_{-T}^T f(t) dt = \frac{1}{T} \int_0^T E_0 dt = E_0$$

$$\begin{aligned} n > 0, \quad a_n &= \frac{1}{T} \int_0^T E_0 \cos \frac{n\pi t}{T} dt = \frac{E_0}{T} \left[\frac{T}{n\pi} \sin \frac{n\pi t}{T} \right]_0^T \\ &= \frac{E_0}{n\pi} (\sin n\pi - 0) = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{T} \int_0^T E_0 \sin \frac{n\pi t}{T} dt = \frac{E_0}{T} \left[-\frac{T}{n\pi} \cos \frac{n\pi t}{T} \right]_0^T \\ &= -\frac{E_0}{n\pi} (\cos n\pi - 1) \end{aligned}$$

$$b_n = \frac{f_0}{n\pi} (1 - (-1)^n) = \begin{cases} \frac{2f_0}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\Rightarrow f(t) = \frac{1}{2}f_0 + \sum_{n \text{ odd}} \frac{2f_0}{n\pi} \sin \frac{n\pi t}{T}$$

$$f = \frac{1}{2}f_0 + \sum_{m=1}^{\infty} \frac{2f_0}{(2m-1)\pi} \sin \left(\frac{(2m-1)\pi t}{T} \right)$$

$$\Rightarrow L \frac{dI}{dt} + RI = \frac{1}{2}f_0 + \sum_{m=1}^{\infty} \frac{2f_0}{(2m-1)\pi} \sin \frac{(2m-1)\pi t}{T}$$

Ans. eqn is $Lm + R = 0 \Rightarrow m = -R/L$

$$\Rightarrow I_{cf} = A e^{-Rt/L}$$

Try $I_{PI} = A_0 + \sum_{m=1}^{\infty} \left\{ A_m \sin \frac{(2m-1)\pi t}{T} + B_m \cos \frac{(2m-1)\pi t}{T} \right\}$

$$\Rightarrow L \frac{dI_{PI}}{dt} + RI_{PI} = L \sum_{m=1}^{\infty} \left\{ \frac{(2m-1)\pi}{T} A_m \cos \frac{(2m-1)\pi t}{T} - \frac{(2m-1)\pi}{T} B_m \sin \frac{(2m-1)\pi t}{T} \right\}$$

$$+ R \left[A_0 + \sum_{m=1}^{\infty} \left\{ A_m \sin \frac{(2m-1)\pi t}{T} + B_m \cos \frac{(2m-1)\pi t}{T} \right\} \right]$$

$$= \frac{1}{2}f_0 + \sum_{m=1}^{\infty} \frac{2f_0}{(2m-1)\pi} \sin \frac{(2m-1)\pi t}{T}$$

$$\Rightarrow \boxed{A_0 = \frac{f_0}{2R}}$$

$$\Rightarrow L \frac{(2n-1)\pi}{T} A_m + R B_m = 0$$

$$-L \frac{(2n-1)\pi}{T} B_m + R A_m = \frac{2E_0}{(2n-1)\pi}$$

$$B_m = -\frac{L(2n-1)\pi}{RT} A_m$$

$$\Rightarrow \frac{L^2(2n-1)^2\pi^2}{RT^2} A_m + R A_m = \frac{2E_0}{(2n-1)\pi}$$

$$\Rightarrow \frac{A_m}{RT^2} \{L^2(2n-1)^2\pi^2 + R^2T^2\} = \frac{2E_0}{(2n-1)\pi}$$

$$\Rightarrow A_m = \frac{2E_0 RT^2}{(2n-1)\pi \{L^2(2n-1)^2\pi^2 + R^2T^2\}}$$

$$B_m = \frac{-2LT E_0}{\cancel{(2n-1)\pi} \{L^2(2n-1)^2\pi^2 + R^2T^2\}}$$

factor $(2n-1)\pi$
was erroneously
included in
recording and
earlier notes

$$\Rightarrow I = A e^{-Rt/L} + E_0/2R$$

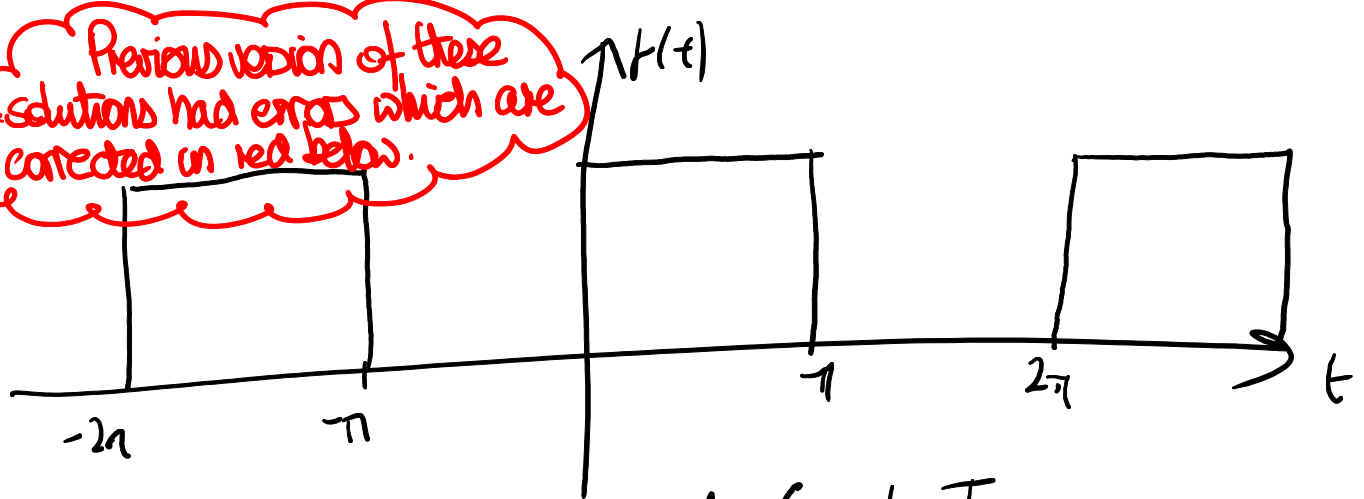
$$+ \sum_{n=1}^{\infty} \frac{2E_0 RT^2}{(2n-1)\pi \{L^2(2n-1)^2\pi^2 + R^2T^2\}}$$

$$\sin \frac{(2n-1)\pi t}{T}$$

$$- \frac{2E_0 LT}{\cancel{(2n-1)\pi} \{L^2(2n-1)^2\pi^2 + R^2T^2\}}$$

$$\cos \frac{(2n-1)\pi t}{T}$$

2) Previous version of these solutions had errors which are corrected in red below.



Same function as Q1, with $E_0 = 1$, $T = \pi$.

$$\Rightarrow f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin(2n-1)t$$

$$y'' + ky' + 25y = f(t)$$

Aux. eqn is $m^2 + km + 25 = 0$

$$\Rightarrow m = \frac{-k \pm \sqrt{k^2 - 100}}{2} = \frac{-k \pm i\sqrt{100 - k^2}}{2}$$

$$\Rightarrow y_{\alpha} = e^{-kt/2} \left(A \cos \frac{\sqrt{100-k^2}}{2} t + B \sin \frac{\sqrt{100-k^2}}{2} t \right)$$

$$\text{Try } y_{\pi} = L + \sum_{n=1}^{\infty} \{ C_n \sin(2n-1)t + D_n \cos(2n-1)t \}$$

$$\Rightarrow y''_{\pi} + ky'_{\pi} + 25y_{\pi} = \sum_{n=1}^{\infty} -(2n-1)^2 \{ C_n \sin(2n-1)t + D_n \cos(2n-1)t \}$$

$$+ k \sum_{n=1}^{\infty} \{ (2n-1) C_n \cos(2n-1)t - (2n-1) D_n \sin(2n-1)t \}$$

$$+ 25 \left[L + \sum_{n=1}^{\infty} \{ C_n \sin(2n-1)t + D_n \cos(2n-1)t \} \right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin(2n-1)t \quad \Rightarrow L = \frac{1}{50}$$

$$(25 - (2n-1)^2)C_n - k(2n-1)D_n = \frac{2}{(2n-1)\pi}$$

$$(25 - (2n-1)^2)D_n + k(2n-1)C_n = 0$$

$$C_n = - \frac{(25 - (2n-1)^2)}{k(2n-1)} D_n$$

$$- \frac{(25 - (2n-1)^2)^2}{k(2n-1)} D_n - k(2n-1)D_n = \frac{2}{(2n-1)\pi}$$

$$\Rightarrow \frac{-\{(25 - (2n-1)^2)^2 + k^2(2n-1)^2\} D_n}{k} = \frac{2}{\pi}$$

$$\Rightarrow D_n = - \frac{2k}{\pi \{(25 - (2n-1)^2)^2 + (2n-1)^2 k^2\}}$$

$$C_n = \frac{2(25 - (2n-1)^2)}{\pi(2n-1) \{(25 - (2n-1)^2)^2 + (2n-1)^2 k^2\}}$$

$$\Rightarrow y = e^{-kt/2} \left\{ A \cos \frac{\sqrt{100-k^2}}{2} t + B \sin \frac{\sqrt{100-k^2}}{2} t \right\}$$

$$+ \frac{1}{50} + \sum_{n=1}^{\infty} C_n \sin(2n-1)t + D_n \cos(2n-1)t$$

When $k=0 \Rightarrow$

$$y = A \cos 5t + B \sin 5t + \frac{1}{50} + \sum_{n=1}^{\infty} \frac{2 \sin(2n-1)t}{\pi(2n-1)(25-(2n-1)^2)}$$

zero when $n=3$
not a valid solution

For $n=3$, try

$$y_{PI} = E t \cos 5t$$

$$\Rightarrow y'_{PI} = E (\cos 5t - 5t \sin 5t)$$

$$y''_{PI} = E (-10 \sin 5t - 25t \cos 5t)$$

$$\Rightarrow y''_{PI} + 25y_{PI} = -10E \sin 5t = \frac{2}{5\pi} \sin 5t$$

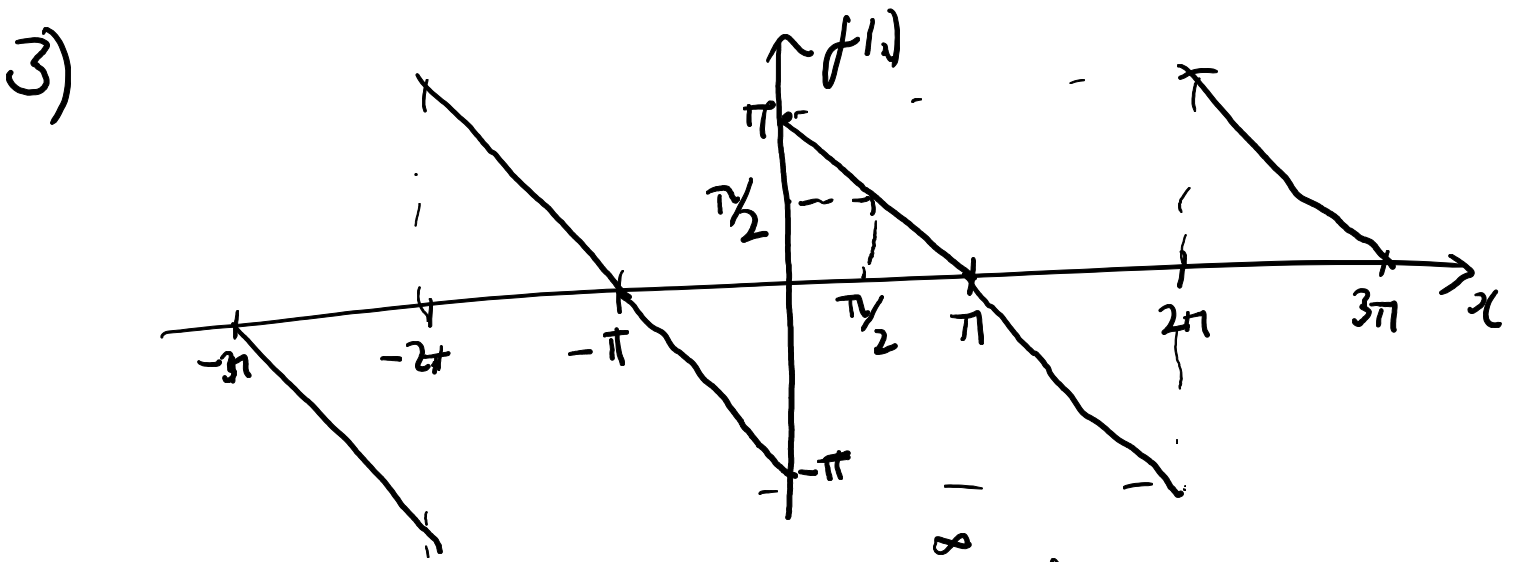
$$\Rightarrow E = -\frac{2}{50\pi}$$

So when $k=0 \Rightarrow$

$$y = A \cos 5t + B \sin 5t + \frac{1}{50} + \sum_{n=1, n \neq 3}^{\infty} \frac{\sin(2n-1)t}{\pi(2n-1)(25-(2n-1)^2)}$$

$$-\frac{2}{50\pi} t \cos 5t$$

resonance!



Odd function $\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[-\frac{1}{n} (\pi - x) \cos nx \right]_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} \cos nx \, dx$$

$$= \frac{2}{\pi} \cdot -\frac{1}{n} (-\pi) - \frac{2}{n\pi} \left[\frac{1}{n} \sin nx \right]_0^{\pi}$$

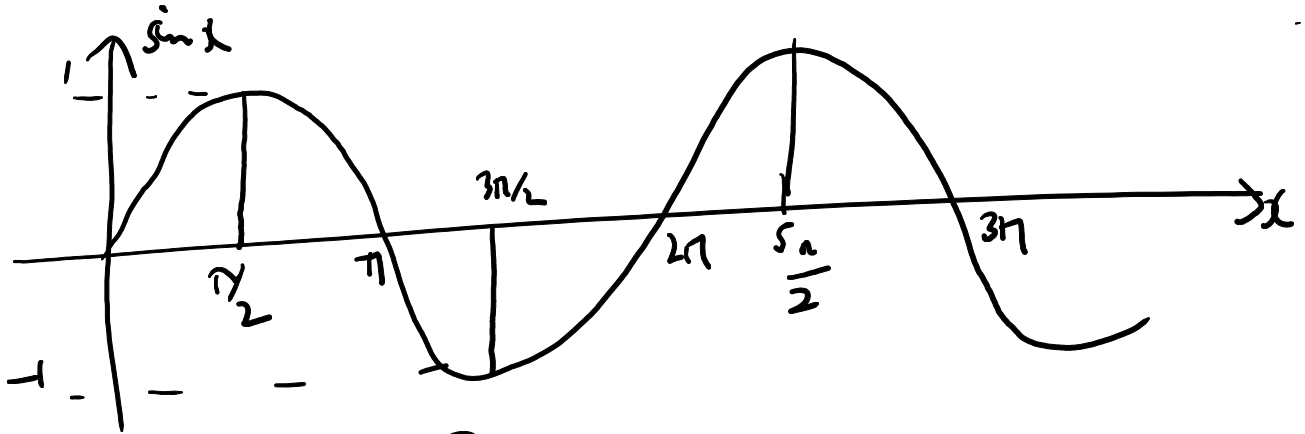
$$= \frac{2}{n}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

Converges to $\pi/2$ at $x = \pi/2$ & 0 at $x = 2\pi$.

At $x = \frac{\pi}{2} \Rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2}{n} \sin \frac{n\pi}{2}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} = \frac{\pi}{4}$$



$$\Rightarrow \sin \frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ 1 & n = 1, 5, 9 \\ -1 & n = 3, 7, 11 \end{cases}$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} = \sum_{n=1}^{\infty} \frac{1}{(2m-1)} \sin \left((m-\frac{1}{2})\pi \right)$$

$$= \sum_{m=1}^{\infty} \frac{1}{(2m-1)} (-1)^{m+1}$$

Let $m = k+1 \Rightarrow \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)}$

$$\frac{d^2 y}{dx^2} + 2y = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

Aux. eqn is $m^2 + 2 = 0 \Rightarrow m = \pm \sqrt{2}$

$$\Rightarrow y_{cf} = A \cos \sqrt{2}x + B \sin \sqrt{2}x$$

$$\text{Try } y_{PI} = \sum_{n=1}^{\infty} B_n \sin nx$$

$$\Rightarrow y_{PI}'' + 2y_{PI} = \sum_{n=1}^{\infty} -n^2 B_n \sin nx + \sum_{n=1}^{\infty} 2 B_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$\Rightarrow (2 - n^2) B_n = \frac{2}{n}$$

$$\Rightarrow B_n = \frac{2}{n(2 - n^2)} \quad \left(\begin{array}{l} \text{Note} \\ 2 - n^2 \neq 0 \end{array} \right)$$

$$\Rightarrow y = A \cos \sqrt{2}x + B \sin \sqrt{2}x + \sum_{n=1}^{\infty} \frac{2}{n(2 - n^2)} \sin nx$$