Problem Sheet 4 – Solutions

1) Remember $\quad L(f(t)) = \int_0^\infty f(t)e^{-st}\,dt = \bar{f}(s)$

$$L(2e^{4t}) = \int_0^\infty 2e^{4t}e^{-st}\,dt = 2\int_0^\infty e^{(4-s)t}\,dt$$

$$= 2\left[\frac{e^{(4-s)t}}{4-s}\right]_0^\infty = 2\left(-\frac{1}{4-s}\right)$$

$$= \frac{2}{s-4} \qquad \text{(Assuming } s>4\text{).}$$

$$L(3e^{-2t}) = \int_0^\infty 3e^{(-2-s)t}\,dt = \left[\frac{3e^{(-2-s)t}}{(-2-s)}\right]_0^\infty$$

$$= 3\left(-\frac{1}{-2-s}\right) = \frac{3}{s+2}$$

$$L(5t-3) = \int_0^\infty (5t-3)e^{-st}\,dt$$

$$= \left[-(5t-3)\frac{e^{-st}}{s}\right]_0^\infty + 5\int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= -\frac{3}{s} + 5\left[-\frac{e^{-st}}{s^2}\right]_0^\infty$$

$$= -\frac{3}{s} + \frac{5}{s^2}$$

$$\mathcal{L}\left(2t^2 - e^{-t}\right) = \int_0^\infty \left(2t^2 - e^{-t}\right)e^{-st}\,dt$$

$$= 2\int_0^\infty t^2 e^{-st}\,dt - \int_0^\infty e^{-(1+s)t}\,dt$$

$$\left[-\frac{e^{-(1+s)t}}{1+s}\right]_0^\infty = \frac{1}{s+1}$$

$$2\Big[\underbrace{\cancel{-\frac{t^2 e^{-st}}{s}}}_{=0}\Big]_0^\infty + \frac{4}{s}\int_0^\infty t e^{-st}\,dt$$

$$= \frac{4}{s}\Big[\underbrace{\cancel{-\frac{t e^{-st}}{s}}}_{=0}\Big]_0^\infty + \frac{4}{s^2}\int_0^\infty e^{-st}\,dt$$

$$= \frac{4}{s^2}\left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{4}{s^3}$$

$$\Rightarrow \mathcal{L}\left(2t^2 - e^{-t}\right) = \frac{4}{s^3} - \frac{1}{s+1}$$

$$\mathcal{L}(3\cos 5t) = 3\int_0^\infty \cos 5t \, e^{-st} \, dt$$

Note $\cos 5t$ is the real part of e^{5it}.

So consider $\displaystyle\int_0^\infty e^{5it} e^{-st} \, dt = \int_0^\infty e^{(5i-s)t} \, dt$

$$= \left[\frac{e^{(5i-s)t}}{5i-s} \right]_0^\infty = \frac{1}{s-5i} = \frac{s+5i}{(s-5i)(s+5i)}$$

$$= \frac{s+5i}{s^2+25} \Rightarrow \mathcal{L}(3\cos 5t) = \frac{3s}{s^2+25}$$

$$\mathcal{L}(10\sin 6t) = \int_0^\infty 10 \sin 6t \, e^{-st} \, dt$$

Note $\sin 6t$ is the imaginary part of e^{6it}

Consider $\displaystyle\int_0^\infty e^{6it} e^{-st} \, dt = \left[\frac{e^{(6i-s)t}}{6i-s} \right]_0^\infty = \frac{1}{s-6i}$

$$= \frac{s+6i}{(s-6i)(s+6i)} = \frac{s+6i}{s^2+36}$$

$$\Rightarrow \mathcal{L}(10\sin 6t) = \frac{60}{s^2+36}$$

2) a) $f(t) = \begin{cases} 0, & 0 \leq t \leq 2 \\ 4, & t > 2 \end{cases} = 4H(t-2)$

$$\Rightarrow \mathcal{L}(f(t)) = 4e^{-2s}/s$$

b) $f(t) = e^{\alpha t} \sinh kt$

We know that

$$\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\mathcal{L}(e^{at} f(t)) = \bar{f}(s-a)$$

$$\mathcal{L}(e^{\alpha t} \sinh kt) = \frac{k}{(s-\alpha)^2 - k^2}$$

3) $\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = te^{3t}$$

first shifting theorem

Note $\mathcal{L}(t) = \frac{1}{s^2}$

Note $\quad \dfrac{a}{s(s+a)} = \dfrac{A}{s} + \dfrac{B}{s+a}$

$$\Rightarrow \quad a = A(s+a) + Bs$$

$s=0 \Rightarrow A=1, \quad s=-a \Rightarrow B=-1,$

$$\Rightarrow \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{a}{s(s+a)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+a}\right)$$

$$= 1 - e^{-at}$$

Note $\quad \dfrac{k^2}{s(s^2+k^2)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+k^2}$

$$\Rightarrow \quad k^2 = A(s^2+k^2) + s(Bs+C)$$

$s=0 \Rightarrow A=1 \qquad \Rightarrow \cancel{k^2} = s^2 + \cancel{k^2} + Bs^2 + Cs$

$$\Rightarrow \quad B=-1, \quad C=0,$$

$$\Rightarrow \frac{k^2}{s(s^2+k^2)} = \frac{1}{s} - \frac{s}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{k^2}{s(s^2+k^2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+k^2}\right)$$

$$= 1 - \cos kt$$

Note $\dfrac{6s-4}{s^2-4s+20} = \dfrac{6s-4}{s^2-4s+4+16} = \dfrac{6s-4}{(s-2)^2+4^2}$

$$= \frac{6(s-2)+8}{(s-2)^2+4^2}$$

note 'b^2-4ac' $= 16 - 4.20 < 0$

$\Rightarrow$ no real factors

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{6s-4}{s^2-4s+20}\right) = 6\mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+4^2}\right) + 2\mathcal{L}^{-1}\left(\frac{4}{(s-2)^2+4^2}\right)$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^4}\right) = \frac{1}{6}(t-2)^3\, H(t-2)$$

second shifting theorem

Note $\mathcal{L}^{-1}\left(\dfrac{6}{s^4}\right) = t^3$

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4) $$\frac{d^2y}{dt^2} - 2a\frac{dy}{dt} + (a^2+b^2)y = 0$$

Take the Laplace transform

$$s^2\bar{y} - s\underbrace{y(0)}_{=0} - \underbrace{\frac{dy}{dt}(0)}_{=1} - 2a\left(s\bar{y} - \underbrace{y(0)}_{=0}\right) + (a^2+b^2)\bar{y} = 0$$

$$\Rightarrow (s^2-2as+a^2+b^2)\bar{y} = 1$$

$$\Rightarrow \bar{y} = \frac{1}{s^2-2as+a^2+b^2} = \frac{1}{(s-a)^2+b^2}$$

$$\Rightarrow y = \frac{e^{at}}{b} \sin bt$$

5) $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 1 - H(t-4)$$

Take Laplace transform $\Rightarrow$

$$s^2\bar{y} - s\cancel{y(0)} - \cancel{\frac{dy}{dt}(0)} + 4\left(s\bar{y} - \cancel{y(0)}\right) + 3\bar{y} = \frac{1}{s} - \frac{e^{-4s}}{s}$$

$$\Rightarrow (s^2+4s+3)\bar{y} = \frac{1-e^{-4s}}{s}$$

$$\Rightarrow \bar{y} = \frac{1-e^{-4s}}{s(s+3)(s+1)}$$

Note

$$\frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$\Rightarrow 1 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1)$$

$s=0 \Rightarrow A = 1/3$, $s=-1 \Rightarrow B = -1/2$

$s=-3 \Rightarrow C = 1/6$

$$\Rightarrow \frac{1}{s(s+1)(s+3)} = \frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1-e^{-4s}}{s(s+1)(s+3)}\right) = \mathcal{L}^{-1}\left(\frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}\right)$$

$$-\mathcal{L}^{-1}\left(e^{-4s}\left(\frac{1}{3s}-\frac{1}{2(s+1)}+\frac{1}{6(s+3)}\right)\right)$$

$$=\frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}-\left\{\frac{1}{3}-\frac{1}{2}e^{-(t-4)}+\frac{1}{6}e^{-3(t-4)}\right\}u(t-4)$$

6) a) Note $\mathcal{L}(\sin t)=\dfrac{1}{s^2+1}$

$$\mathcal{L}\left(e^{-2t}f(t)\right)=\bar{f}(s+2)$$

$$\Rightarrow \mathcal{L}\left(e^{-2t}\sin t\right)=\frac{1}{(s+2)^2+1}=\frac{1}{s^2+4s+5}$$

b) $$\mathcal{L}\left\{\frac{1}{5}\left(1-e^{-2t}\cos t-2e^{-2t}\sin t\right)\right\}$$

$$=\frac{1}{5}\left\{\frac{1}{s}-\frac{(s+2)}{(s+2)^2+1}-2\frac{1}{(s+2)^2+1}\right\}$$

$$=\frac{1}{5}\left\{\frac{1}{s}-\frac{s+4}{s^2+4s+5}\right\}$$

$$=\frac{1}{5}\left\{\frac{\cancel{s^2}+\cancel{4s}+5-s(\cancel{s}+\cancel{4})}{s(s^2+4s+5)}\right\}=\frac{1}{s(s^2+4s+5)}$$

c) By the second shifting theorem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = u(t-3)$$

Take Laplace transform

$$\Rightarrow s^2\bar{y} - s\cancel{y(0)} - \underbrace{\frac{dy}{dt}(0)}_{=1} + 4\left(s\bar{y} - \cancel{y(0)}\right) + 5\bar{y} = \frac{e^{-3s}}{s}$$

$$\Rightarrow (s^2+4s+5)\bar{y} = 1 + \frac{e^{-3s}}{s}$$

$$\Rightarrow \bar{y} = \frac{1}{s^2+4s+5} + \frac{e^{-3s}}{s(s^2+4s+5)}$$

$$\Rightarrow y = e^{-2t}\sin t + \frac{1}{5}\left(1 - e^{-2(t-3)}\cos(t-3) - 2e^{-2(t-3)}\sin(t-3)\right)u(t-3)$$