Problem Sheet 5 – Solutions

1) $$\frac{\partial^2 \phi}{\partial t \partial x} + x e^t \phi = 0$$

Look for a solution $\phi = T(t) X(x)$

$$\Rightarrow T'X' + x e^t T X = 0 \Rightarrow T'X' = -x e^t TX$$

$$\Rightarrow \underbrace{\frac{T'}{T} e^{-t}}_{\text{fn of } t \text{ only}} = \underbrace{-\frac{xX}{X'}}_{\text{fn of } x \text{ only}} = \underbrace{c}_{\text{constant}}$$

$\Rightarrow$ constant.

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = c e^t \Rightarrow \log T = c e^t + \text{const.}$$

$$\Rightarrow \boxed{T = A e^{c e^t}}$$

$$\Rightarrow \frac{1}{X}\frac{dX}{dx} = -\frac{x}{c} \Rightarrow \log X = -\frac{x^2}{2c} + \text{const.}$$

$$\Rightarrow \boxed{X = B e^{-x^2/2c}}$$

$$\Rightarrow \boxed{\phi = XT = D e^{c e^t - x^2/2c}}$$

2) $\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$ for $0<x<L$, $T(0,t) = T_1$
$T(L,t) = T_2$

a) If $T = T_s(x) \Rightarrow 0 = D\frac{d^2 T_s}{dx^2}$

$\Rightarrow T_s = ax + b$

$T_s(0) = T_1 = b \Rightarrow b = T_1$

$T_s(L) = aL + T_1 = T_2 \Rightarrow a = \frac{T_2 - T_1}{L}$

$$\Rightarrow \boxed{T_s(x) = \frac{(T_2 - T_1)}{L}x + T_1}$$

b) $T(x,0) = T_1$.

Let $\bar{T}(x,t) = T(x,t) - T_s(x) \Rightarrow T(x,t) = \bar{T} + T_s(x)$

So $\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\partial \bar{T}}{\partial t} = D\left(\frac{\partial^2 \bar{T}}{\partial x^2} + \underbrace{\cancel{\frac{d^2 T_s}{dx^2}}}_{=0}\right)$

$$\Rightarrow \boxed{\frac{\partial \bar{T}}{\partial t} = D\frac{\partial^2 \bar{T}}{\partial x^2}}$$

$T(0,t) = T_1 = \bar{T}(0,t) + T_s(0)$
$= \bar{T}(0,t) + T_1$

$$\Rightarrow \boxed{\bar{T}(0,t) = 0}$$

$T(L,t) = T_2 = \bar{T}(L,t) + T_s(L) = \bar{T}(L,t) + T_2$

$$\Rightarrow \boxed{\bar{T}(L,t) = 0}$$

$$T(x,0) = T_1 = \bar{T}(x,0) + T_s(x)$$

$$\Rightarrow \bar{T}(x,0) = -\frac{(T_2 - T_1)}{L}x + T_1 \quad -T_1$$

There were sign errors in the original version of these solutions which are corrected in red below

$$\Rightarrow \boxed{\bar{T}(x,0) = -\left(\frac{T_2 - T_1}{L}\right)x}$$

Seek a separable solution

$$\bar{T}(x,t) = X(x)\tau(t)$$

$$\Rightarrow X\tau' = DX''\tau$$

$$\Rightarrow \underbrace{\frac{\tau'}{\tau}}_{\text{fn of } t \text{ only}} = \underbrace{\frac{DX''}{X}}_{\text{fn of } x \text{ only}} = -c \Rightarrow \text{constant}$$

$$\Rightarrow \tau' = -c\tau \Rightarrow \tau = \bar{A}e^{-ct} \Rightarrow c > 0$$

for decaying solutions.

$$\& \quad DX'' + cX = 0$$

$$\Rightarrow X = A \sin\sqrt{\frac{c}{D}}\,x + B\cos\sqrt{\frac{c}{D}}\,x$$

$$X(0) = 0 = B$$

$$X(L) = 0 = A\sin\sqrt{\frac{c}{D}}\,L$$

$$\Rightarrow \sqrt{\frac{c}{D}}\,L = n\pi, \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow c = \frac{n^2\pi^2 D}{L^2}$$

$$\Rightarrow X = A\sin\frac{n\pi x}{L}, \quad \tau = \bar{A}\, e^{-n^2\pi^2 D t/L^2}$$

$$\Rightarrow \boxed{\bar{T}(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\, e^{-n^2\pi^2 D t/L^2}}$$

As $t\to\infty$, $\bar{T}\to 0 \Rightarrow T \to T_s(x)$.

$$\bar{T}(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = -\left(\frac{T_2 - T_1}{L}\right)x$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L -\left(\frac{T_2 - T_1}{L}\right)x \sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{2(T_2 - T_1)}{L^2}\left[\left[-\frac{L}{n\pi}\,x\cos\frac{n\pi x}{L}\right]_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right]$$

$$= -\frac{2(T_2 - T_1)}{L^2}\left[-\frac{L^2}{n\pi}\cos n\pi + \frac{L}{n\pi}\left[\frac{L}{n\pi}\underbrace{\sin\frac{n\pi x}{L}}_{=0}\right]_0^L\right]$$

$$= +\frac{2(T_2 - T_1)}{n\pi}(-1)^n$$

$$\Rightarrow \boxed{T(x,t) = T_s(x) + \frac{2(T_2-T_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L}\, e^{-n^2\pi^2 Dt/L^2}}$$

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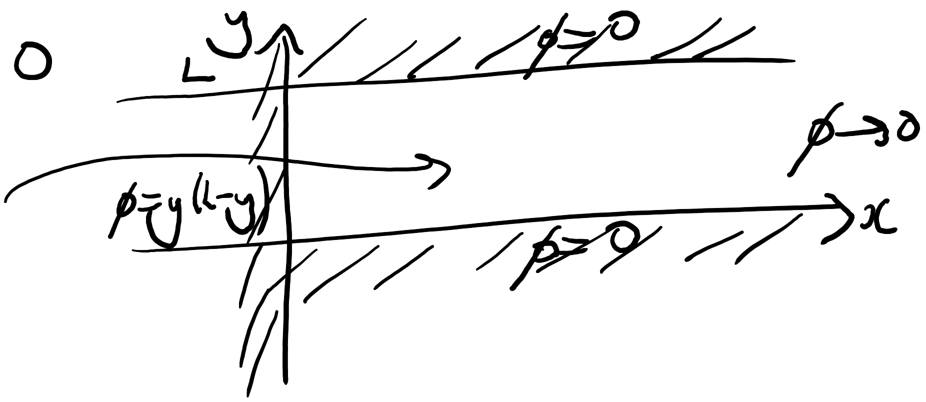
3) $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$



$\phi = 0$ at $y = 0$

$\phi = 0$ at $y = L$

$\phi = y(L-y)$ at $x = 0$

$\phi \to 0$ as $x \to \infty$

Seek a separable solution $\phi = X(x)Y(y)$

$$\Rightarrow X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = c$$

fn of x only, fn of y only $\Rightarrow$ constant

$$\Rightarrow X'' - cX = 0$$

If $c=0 \Rightarrow X''=0$ & $X \to 0$ as $x \to \infty$ $\Rightarrow X=0$
trivial

If $c>0 \Rightarrow X = Ae^{c^{1/2}x} + Be^{-c^{1/2}x}$

& $X \to 0$ as $x \to \infty$ $\Rightarrow$ $A=0$

$\Rightarrow X = Be^{-c^{1/2}x}$ $\quad Y'' + cY = 0$

$\Rightarrow Y = D \sin c^{1/2}y + E \cos c^{1/2}y$

$Y(0) = E = 0$

$Y(L) = D \sin c^{1/2}L = 0 \quad \Rightarrow c^{1/2}L = n\pi$

$n = 1, 2, 3, \ldots$

$\Rightarrow c = n^2\pi^2/L^2$

$\Rightarrow Y \propto \sin \frac{n\pi y}{L}, \quad X \propto e^{-n\pi x/L}$

$$\Rightarrow \boxed{\text{general solution is} \quad \phi = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{L} e^{-n\pi x/L}}$$

If $c<0$, $X'' - cX = 0$

$\Rightarrow X = \bar{A} \sin(-c)^{1/2}x + \bar{B} \cos(-c)^{1/2}x$

$X \to 0$ as $x \to \infty$ $\Rightarrow X=0$, trivial.

$$\phi(0,y) = y(L-y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{L}$$

Another odd Fourier series

$$\Rightarrow \quad A_n = \frac{2}{L}\int_0^L y(L-y)\sin\frac{n\pi y}{L}\,dy$$

$$= \frac{2}{L}\left[\cancelto{0}{-y(L-y)\frac{L}{n\pi}\cos\frac{n\pi y}{L}}\right]_0^L + \frac{2}{L}\int_0^L (L-2y)\frac{L}{n\pi}\cos\frac{n\pi y}{L}\,dy$$

$$= \frac{2}{n\pi}\left[\cancelto{0}{(L-2y)\frac{L}{n\pi}\sin\frac{n\pi y}{L}}\right]_0^L + \frac{4L}{(n\pi)^2}\int_0^L \sin\frac{n\pi y}{L}\,dy$$

$$= \frac{4L}{(n\pi)^2}\left[-\frac{L}{n\pi}\cos\frac{n\pi y}{L}\right]_0^L$$

Original version had a propagating error, which is corrected in red

$$= -\frac{4L^2}{n^3\pi^3}\left(\cos n\pi - 1\right) = \frac{4L^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$= \begin{cases} 0, & n \text{ even} \\ 8L^2/n^3\pi^3, & n \text{ odd} \end{cases}$$

$$\boxed{\phi = \sum_{m=1}^{\infty} \frac{8L^2}{(2m-1)^3\pi^3}\sin\frac{(2m-1)\pi y}{L}\, e^{-(2m-1)\pi x/L}}$$

4) $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g \sin \omega t, \quad y(0,t) = y(L,t) = 0$$

a) Let $y = \bar{Y}(x) \sin \omega t$

$$\Rightarrow -\omega^2 \bar{Y} \sin \omega t = c^2 \bar{Y}'' \sin \omega t + g \sin \omega t$$

$$\Rightarrow c^2 \bar{Y}'' + \omega^2 \bar{Y} = -g$$

$$\Rightarrow \bar{Y} = A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c} - g/\omega^2$$

$$\bar{Y}(0) = B - g/\omega^2 = 0 \Rightarrow B = g/\omega^2$$

$$\bar{Y}(L) = A \sin \omega L/c + g/\omega^2 \cos \frac{\omega L}{c} - g/\omega^2 = 0$$

$$\Rightarrow A = \frac{g}{\omega^2} \frac{(1 - \cos \omega L/c)}{\sin \omega L/c}$$

$$\Rightarrow \bar{Y} = \frac{g}{\omega^2} \left[\frac{(1 - \cos \omega L/c)}{\sin \omega L/c} \sin \omega x/c + \cos \omega x/c - 1 \right]$$

$$\Rightarrow y_f = \frac{g}{\omega^2} \left[\frac{(1 - \cos \omega L/c)}{\sin \omega L/c} \sin \frac{\omega x}{c} + \cos \frac{\omega x}{c} - 1 \right] \sin \omega t$$

Resonance when $\omega L/c = n\pi \Rightarrow \omega = n\pi c/L$.

d) find solution with $y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0$.

Let $y = y_f + \hat{y}(x,t)$

so $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g \sin \omega t$$

$$\frac{\partial^2 y_f}{\partial t^2} = c^2 \frac{\partial^2 y_f}{\partial x^2} + g \sin \omega t$$

subtract $\Rightarrow$ $\boxed{\frac{\partial^2 \hat{y}}{\partial t^2} = c^2 \frac{\partial^2 \hat{y}}{\partial x^2}}$, $\hat{y}(0,t) = \hat{y}(L,t) = 0$

$$y(x,0) = y_f(x,0) + \hat{y}(x,0)$$

$\Rightarrow 0 = 0 + \hat{y}(x,0) \Rightarrow \boxed{\hat{y}(x,0) = 0}$

& $$\frac{\partial y}{\partial t}(x,0) = \frac{\partial y_f}{\partial t}(x,0) + \frac{\partial \hat{y}}{\partial t}(x,0)$$

$\Rightarrow 0 = \omega \bar{Y}(x) + \frac{\partial \hat{y}}{\partial t}(x,0)$

$\Rightarrow \boxed{\frac{\partial \hat{y}}{\partial t}(x,0) = -\omega \bar{Y}(x)}$

Seek a separable solution $\hat{y} = X(x)T(t)$

$$XT'' = c^2 X'' T \Rightarrow \frac{T''}{T} = c^2 \frac{X''}{X} = -k^2$$

fn of t only fn of x only $\Rightarrow$ constant

$$\Rightarrow T = A\sin kt + B\cos kt$$

$$X = C\sin\frac{k}{c}x + D\cos\frac{k}{c}x$$

$$X(0) = 0 \Rightarrow D = 0$$

$$X(L) = 0 \Rightarrow \sin\frac{kL}{c} = 0 \Rightarrow k = \frac{n\pi c}{L}, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \hat{Y}(x,t) = \sum_{n=1}^{\infty}\left\{A_n \sin\frac{n\pi ct}{L} + B_n\cos\frac{n\pi ct}{L}\right\}\sin\frac{n\pi x}{L}$$

$$\hat{Y}(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = 0 \Rightarrow B_n = 0$$

$$\Rightarrow \hat{Y}(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$$

$$\Rightarrow \frac{\partial \hat{Y}}{\partial t}(x,0) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}A_n\sin\frac{n\pi x}{L} = -w\bar{Y}(x)$$

$$\Rightarrow \frac{n\pi c}{L}A_n = \frac{2}{L}\int_0^L -w\bar{Y}(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow \boxed{A_n = \frac{2w}{n\pi c}\int_0^L \bar{Y}(x)\sin\frac{n\pi x}{L}\,dx}$$