MTHS2007 Problem Sheet 6 Solutions

Q1 (a) $P(A \cup B) = P(A) + P(B) = 0.3 + 0.17 = 0.49$

(b) $P(C^c) = 1 - P(C) = 1 - 0.21 = 0.79$

(c) $P(B \cup D) = P(B) + P(D) = 1 - P(A) - P(C)$
$= 1 - 0.3 - 0.21$
$= 0.49.$

(Using the sum rule and the fact that all relevant events are disjoint)

Q2 (a) $P(\text{Blue}) = \dfrac{\#\text{Blue}}{\#\text{blocks}} = \dfrac{12}{30} = 0.4.$

(b) i) $P(\text{Yellow}, \text{Yellow}) = P(\text{Yellow})\, P(\text{Yellow} \mid \text{Yellow})$
$= P(\text{Yellow})\, P(\text{Yellow})$ since the block is replaced the outcome of the second draw is independent of the first.
$= \dfrac{8}{30} \times \dfrac{8}{30} = \dfrac{16}{225} \approx 0.0711$

ii) $P(\text{one Yellow + one Red}) = P(\text{Yellow}, \text{Red}) + P(\text{Red}, \text{Yellow})$
$= \dfrac{8}{30} \times \dfrac{10}{30} + \dfrac{10}{30} \times \dfrac{8}{30}$
$= 8/45 \approx 0.1777$

iii) $P(\text{both the same}) = P(Y,Y) + P(R,R) + P(B,B)$
$= \dfrac{8}{30} \times \dfrac{8}{30} + \dfrac{10}{30} \cdot \dfrac{10}{30} + \dfrac{12}{30} \cdot \dfrac{12}{30}$
$= 308/900 \approx 0.3422$

iv) $P(\text{different}) = 1 - P(\text{same}) = 1 - \dfrac{308}{900} = \dfrac{592}{900} \approx 0.6578.$

Q2 (c) i) $P(Y,Y) = P(Y)\,P(Y|Y)$ (product rule)

$$= \frac{8}{30} \times \frac{7}{29} = \frac{28}{435} \approx 0.06437$$

ii) $P(R,Y) + P(Y,R) = \frac{8}{30} \cdot \frac{10}{29} + \frac{10}{30} \cdot \frac{8}{29} = \frac{16}{87} \approx 0.1839$

iii) $P(Y,Y) + P(R,R) + P(B,B) = \frac{8}{30} \times \frac{7}{29} + \frac{10}{30} \times \frac{9}{29} + \frac{12}{30} \times \frac{11}{29}$

$$= \frac{278}{870} \approx 0.3195$$

iv) $P(\text{different}) = 1 - P(\text{same}) = 1 - \frac{278}{870} = \frac{592}{870} \approx 0.6805.$

Q3 (a) $P(\text{Pass } 1 \cup \text{Pass } 2) = P(\text{Pass } 1) + P(\text{Pass } 2) - P(\text{Pass both})$

$$= 0.75 + 0.6 - 0.5$$

$$= 0.85.$$

(b) $P(\text{Fail both}) = 1 - P(\text{Pass at least one})$

$$= 1 - 0.85 = 0.15.$$

(c) $P(\text{Pass } 2 \mid \text{Pass } 1) = \frac{P(\text{Pass } 2 \cap \text{Pass } 1)}{P(\text{Pass } 1)} = \frac{0.5}{0.75} = \frac{2}{3}.$

(d) $P(\text{Pass } 2 \mid \text{Fail } 1) = \frac{P(\text{Pass } 2 \cap \text{Fail } 1)}{P(\text{Fail } 1)}$

Now, $P(\text{Pass } 2) = P(\text{Pass } 2 \cap \text{Pass } 1) + P(\text{Pass } 2 \cap \text{Fail } 1)$ (Law of total Prob)

so $0.6 = 0.5 + P(\text{Pass } 2 \cap \text{Fail } 1)$

$\Rightarrow 0.1 = P(\text{Pass } 2 \cap \text{Fail } 1)$

Also $P(\text{Fail } 1) = 1 - P(\text{Pass } 1) = 1 - 0.75 = 0.25$

So $P(\text{Pass } 2 \mid \text{Fail } 1) = \frac{0.1}{0.25} = 0.4.$

Q4 Write $A =$ Acceptable, $X =$ made by machine X
$Y =$ " " " Y.

(a) $P(A|X) = 0.98$.

(b) $P(A^c|Y) = 1 - P(A|Y) = 1 - 0.95 = 0.05$.

(c) $P(A) = P(X)P(A|X) + P(Y)P(A|Y)$ (Law of total prob.)
$= 0.65 \times 0.98 + 0.35 \times 0.95$
$= 0.9695$.

(d) $$P(X|A) = \frac{P(A \cap X)}{P(A)} = \frac{P(X)P(A|X)}{P(A)} = \frac{0.65 \times 0.98}{0.9695}$$
≈ 0.6570.

Q5 Write $X = \#$ acceptable, so $X \sim B(10, 0.92)$.

(a) $P(X=10) = \binom{10}{10} \times 0.92^{10} \times 0.08^0 = 0.92^{10} \approx 0.4344$.

(b) $P(X=8) = \binom{10}{8} \times 0.92^8 \times 0.08^2 \approx 0.1478$.

(c) $P(X \geq 8) = P(X=8) + P(X=9) + P(X=10)$

$= 0.1478 + \binom{10}{9} 0.92^9\, 0.08^1 + 0.4344$

≈ 0.9599.

(d) $E(X) =$ mean $= np = 10 \times 0.92 = 9.2$.

Q6 Write $X = \#$ defective, so $X \sim \text{Bin}(8, 0.2)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 1/5$.

$$P(X=2) = \binom{8}{2}(0.2)^2(0.8)^6 \approx 0.2936$$

$$P(X \geq 6) = \binom{8}{6}(0.2)^6(0.8)^2 + \binom{8}{7}(0.2)^7(0.8)^1 + \binom{8}{8}(0.2)^8(0.8)^0$$

$$\approx 0.00114688 + 0.00008192 + 0.00000256$$

$$\approx 0.001231, \text{ (to 4 s.f.)}$$

Q7 Given that mean $= np = 2$, var $= np(1-p) = \frac{3}{2}$.

First, $\frac{np(1-p)}{np} = 1-p = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$, i.e. $p = 1/4$,

and $np = \frac{n}{4} = 2$, so $n = 8$.

So $P(\text{Bin}(8, 1/4) = 1) = \binom{8}{1}(0.25)^1(0.75)^7$

$$\approx 0.2670.$$

Q8 Write X = # defective in a box, so Poisson mean is

$$\lambda = np = 250 \times \frac{2}{100} = 5 \text{ per box.}$$

$$\left(\text{so } P(X=i) = \frac{e^{-5}5^i}{i!}, \quad i=0,1,2,\dots\right)$$

(a) $P(X=0) = e^{-5} \approx 0.006738$

(b) $P(X=2) = \dfrac{e^{-5}5^2}{2!} \approx 0.08422$

(c) $P(X>1) = 1 - P(X=0) - P(X=1)$

$$= 1 - e^{-5} - e^{-5}.5$$

$$= 1 - 6e^{-5} \approx 0.9596.$$

Q9 Write Y = # releases in a month, so $Y \sim Poi(2)$.

(a) $P(Y \leq 4) = e^{-2}\left(1 + 2 + \dfrac{2^2}{2} + \dfrac{2^3}{6} + \dfrac{2^4}{24}\right)$

$$= 7e^{-2} \approx 0.9473$$

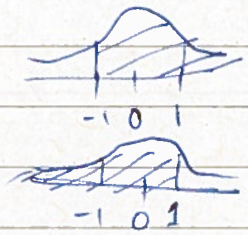
(b) Z = # releases over 3 months

$= Y_1 + Y_2 + Y_3$ where Y_i are independent $Po(2)$ variables.

$$E[Z] = E[Y_1] + E[Y_2] + E[Y_3]$$
$$= 2 + 2 + 2$$
$$= 6.$$

Q10 If $X \sim N(12, 2^2)$ then

(a) $P(X > 10) = P\left(\frac{X-\mu}{\sigma} > \frac{10-12}{2}\right) = P(Z > -1)$



$= P(Z < 1)$

$= 0.8413$

(b) $P(11.4 < X < 14.2) = P\left(\frac{11.4-12}{2} < Z < \frac{14.2-12}{2}\right)$

$= P(-0.3 < Z < 1.1)$

$= P(Z < 1.1) - P(Z < -0.3)$

$= P(Z < 1.1) - (1 - P(Z < 0.3))$

$= 0.8643 - 1 + 0.6179$

$= 0.4822$

(c) $P(9.9 < X < 13.8) = P\left(\frac{9.9-12}{2} < Z < \frac{13.8-12}{2}\right)$

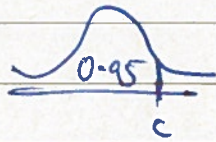
$= P(Z < 0.6) - P(Z < -1.05)$

$= P(Z < 0.6) - (1 - P(Z < 1.05))$

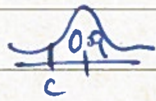
$= 0.7257 - 1 + 0.8531$

$= 0.5788.$

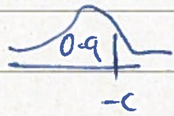
Q11 (a) Directly from tables, $c = 1.645$.



(b) $P(Z \geq c) = 0.9$

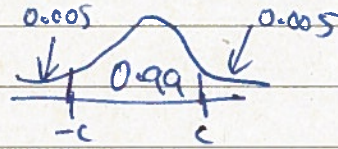


$\iff P(Z < -c) = 0.9$



so from tables $-c = 1.282 \Rightarrow c = -1.282$.

(c) $P(-c \leq Z \leq c) = 0.99$



$\Rightarrow P(Z \leq c) = 0.995$.

so $c = 2.576$.

Q12 Write X = sack weight, so $X \sim N(51.8, \sigma^2)$

Given that $P(X > 52.7) = 0.1$.

This implies that $P\left(Z > \frac{52.7 - 51.8}{\sigma}\right) = 0.1$

i.e. $P\left(Z > \frac{0.9}{\sigma}\right) = 0.1$

or $P\left(Z < \frac{0.9}{\sigma}\right) = 0.9$

From tables this means that $\frac{0.9}{\sigma} = 1.282$,

so $\sigma = \frac{0.9}{1.282} \approx 0.7020$.

Q13 Write M = exam mark of an individual, so $M \sim N(60, 12^2)$.

(a) $P(M > 50) = P\left(\frac{M-\mu}{\sigma} > \frac{50-60}{12}\right) = P(Z > -0.83)$

$= P(Z < 0.83)$

$= 0.7967.$

So on average expect $500 \times 0.7967 = 398.35$, just under 400 candidates to pass.

(b) $P(M>90) = P\left(Z > \frac{90-60}{12}\right) = P(Z > 2.5) = 1 - P(Z<2.5)$

$= 1 - 0.9938 = 0.0062$

So $500 \times 0.0062 = 3.1$; about 3 should score over 90.

(c) Want to find x, the pass mark, so that

$P(M > x) = \frac{450}{500} = 0.9,$

i.e. $P\left(Z > \frac{x-60}{12}\right) = 0.9.$

But since $P(Z<1.282) = 0.9$, $P(Z > -1.282) = 0.9.$

So we need $\frac{x-60}{12} = -1.282$

$\Rightarrow x = 60 + 12 \times (-1.282) = \underline{\underline{44.616}}$

Q14 Call $S =$ std lifetime, $\sim N(1400, 240^2)$
$L =$ long lifetime, $\sim N(1700, 320^2)$.

(a) $P(S > 1000) = P\left(\frac{S-\mu}{\sigma} > \frac{1000-1400}{240}\right) = P(Z > 1.67)$

$= 1 - P(Z < 1.67) = 1 - 0.9525 = 0.0475.$

(b) Call the difference $D = L - S \sim N(\underbrace{1700-1400}_{300}, \underbrace{240^2+320^2}_{400^2})$

So $P(S > L) = P(D < 0) = P\left(\frac{D-\mu}{\sigma} < \frac{0-300}{400}\right)$

$= P(Z > 0.75) = 1 - P(Z < 0.75) = 0.2266.$

(c) $P(L > S + 800) = P(L - S > 800) = P(D > 800).$

Now $P(D > 800) = P\left(Z > \frac{800-300}{400}\right) = P(Z > 1.25)$

$= 1 - P(Z < 1.25) = 1 - 0.8944 = 0.1056.$

Q15 (a) $\bar{x} = \sum_i x_i / n = 170.8/10 = 17.08$

(b) CI Endpoints are $\bar{x} \pm c\frac{\sigma}{\sqrt{n}}$, $\sigma = \sqrt{150}$, $n = 10$.

90% CI so $P(-c < Z < c) = 0.90$

$\Rightarrow P(Z < c) = 0.95$

$\Rightarrow c = 1.645$.

Now $17.05 \pm 1.645 \times \frac{\sqrt{150}}{\sqrt{10}} \approx 10.7$ & 23.4,

So the 90% CI for the mean is between 10.7 and 23.4.

(c) If we want the width less than 5

we need $1.645 \times \frac{\sqrt{150}}{\sqrt{n}} < 2.5$,

i.e. $8.058\cdots < \sqrt{n}$

$\Rightarrow n > 64.94$, so n at least 65.

Q16 (a) Have $\bar{x} = 14.193$, $s = 0.327$, $n = 35$.

$n = 35$ is 'large' (larger than 30) so use the CI formula.

$\bar{x} \pm c \cdot \frac{s}{\sqrt{n}}$, $P(|Z| < c) = 0.95 \Rightarrow c = 1.96$.

So endpoints are $14.193 \pm 1.96 \times \frac{0.327}{\sqrt{35}} = 15.03, 14.80$.

The 95% CI for the mean filling time is $(14.80, 15.03)$.

(b) No further assumptions required.

(Using s not σ; and no assumption of Normality OK since n is 'large').

(c) The claim is not conclusively shown by the data, since values above 15 sec are in the CI.

But it is probably true since most of the plausible values for the mean are below 15 seconds.